PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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A New Determination of the Ratio of the Imperial Standard Yard to the International Prolotype Metre

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VII. *A New Determination of the Ratio of the Imperial Standard Yard to the International Prototype Metre.*

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Abstract.

The method employed for the determination of the ratio involves the use of a subsidiary line standard which is specially graduated in main intervals of $1\frac{1}{8}$ inches. The graduations define a number of approximate yard and metre lengths, 32 main intervals being closely equivalent to one yard and 35 to one metre. The relative values of the yard and metre lengths can be determined by a calibration of the main intervals, while their absolute values are determinable by comparison with the corresponding primary standards or attested copies.

The paper describes how these determinations have been carried out and applied. They result in the following value of the ratio:—

1 metre = 39·370147 inches.

Method briefly outlined.

The method employed for the determination of the ratio of the yard to the metre is due to J. E. SEARS, who conceived the idea of using a line standard graduated in such a way that the relation between certain intervals on it, closely approximating to yard and metre lengths, could be determined by means of an accurate direct calibration of the main subdivisions of the bar, and that subsequently these intervals could be referred either to the primary standards themselves or to well-determined copies. A description of the bar will help to make this clear.

Line Standard Nickel 184.

The bar was specially ruled for the purpose by la Société Genevoise, and is known, from its maker's serial number and the material of which it is made, as Nickel 184, or, briefly, Ni 184. It is made in the usual H section form, with the scale graduated on a

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polished strip in its neutral plane. It is divided in main intervals of $1\frac{1}{8}$ inches for a length of $40\frac{1}{2}$ inches from $-\frac{1}{4}$ inch to $40\frac{1}{4}$ inches, and is further subdivided throughout in $\frac{1}{8}$-inch intervals. Each main graduation defining the $1\frac{1}{8}$-inch intervals is also the centre of a symmetrical group of further subdivisions, consisting of $\frac{1}{40}$-inch and $\frac{1}{200}$-inch intervals. These localised groups of small intervals serve to enable an observer readily to recognise the main graduations.

The main $1\frac{1}{8}$-inch intervals are for convenience referred to as "spans," and it will readily be seen that there are 36 such spans between the lines $-\frac{1}{4}$ inch and $40\frac{1}{4}$ inches. Moreover, 32 spans are equivalent to 36 inches or one yard, while 35 of them are equivalent to 39·375 inches, a length closely equal to 1 metre, the difference of approximately 0·004 inch being conveniently measurable by the direct use of an ordinary comparator microscope with micrometer eyepiece. There are five such yard lengths to be found among the main graduations, 0/32, 1/33, 2/34, 3/35 and 4/36 spans, and two such metre lengths 0/35 and 1/36 spans. Thus the one bar bears both yard and metre intervals, whose relative lengths may be found by a simple calibration of the spans. Further, the one-yard intervals may be taken as approximately equivalent to 32/35 of a metre interval, or, more precisely,

$$Y = 32/35\ M + e, \quad \ldots \ldots \ldots \ldots \ldots \quad (1)$$

where Y is any one of the yard intervals, M one of the metre intervals, and e a small correction.

Main Divisions of the Observational Work.

The equation (1) contains three unknowns, Y, M and e, and the observational work naturally falls into three main divisions corresponding to the independent determinations of these quantities. The divisions may briefly be stated as

(i) A calibration of the "spans," so as to include particularly those graduations pertaining to the yard and metre intervals. This determines e.

(ii) The comparison of one or more yard intervals with the Imperial Standard Yard.

(iii) The comparison of one or both of the metre intervals with the International Prototype metre or any well-verified copy of same.

Details of the manner in which these three steps of the work have been accomplished are given in the following paragraphs.

Calibration of the Spans.

The method of carrying out the observations and of computing the results for a calibration of the subdivisions of a line standard is too well known to need more than a brief mention here; reference may be made to GLAZEBROOK'S 'Dictionary of Applied Physics' (Macmillan), vol. 3, article on "Line Standards," for more complete details.

In the present case the defining lines of the yard and metre intervals referred to above are all included in the first and last six spans of the bar, and this has an important bearing on the scheme of calibration. It was, in fact, possible to calibrate the scale in six entirely different ways, each including at least one of the principal metre lengths, and four of the principal yard lengths, the six resulting independent calibrations being finally combined in one adjusted mean calibration.

Each calibration consists of two parts,

(*a*) A main calibration, involving the intercomparison of the intervals resulting from the division of the whole scale, or such part of it as may be selected, into a number of equal parts, each of which is a multiple of $1\frac{1}{8}$ inch (the length of a span); and

(*b*) A subsidiary calibration involving a comparison of the spans in the first interval of the main calibration with those of the last interval, these two intervals being the ones involving some or all of the important graduations defining the metre and yard lengths.

For example, the first of the six calibrations carried out involves (*a*) an intercomparison of six intervals each 6 spans long, and (*b*) a comparison of the six spans in the first interval with the six spans in the last. Similarly, the second calibration consists of (*a*) nine main intervals each four spans long, and (*b*) a comparison of the first four spans with the last four. These may be symbolically expressed thus, 6×6 and 9×4, the first number of each pair representing the main calibration. Both these calibrations cover the whole range 0/36 spans of the scale, including both metre intervals. The remaining calibrations cover only 35 spans of the whole scale. Two of them are calibrations of one of the main metre intervals, 0/35 spans, and consist of 7×5 and 5×7 combinations. The other two are exactly similar calibrations for the other main metre interval 1/36 spans. Fig. 1, where these various calibrations are shown graphically, will help to make the preceding explanation clear.

In the second calibration referred to above, the number of intervals in the subsidiary portion is only 4, and the number of observations correspondingly small, the latter in a calibration of this kind varying as the square of the number of intervals. In order, therefore, to increase the weight of the results in the final computation, the whole of calibration 2 was done twice.

The observational work in connection with the calibration was carried out solely by H. L. P. JOLLY, who was responsible also for the subsequent computation of the final calibration. This computation is necessarily a complicated one, since different weights have to be given to the various values resulting from each calibration in accordance with the number of observations involved; and also the end points for each calibration not being the same throughout must be adjusted amongst themselves. The method devised by JOLLY for carrying out this part of the work is given in detail in the Appendix to this paper.

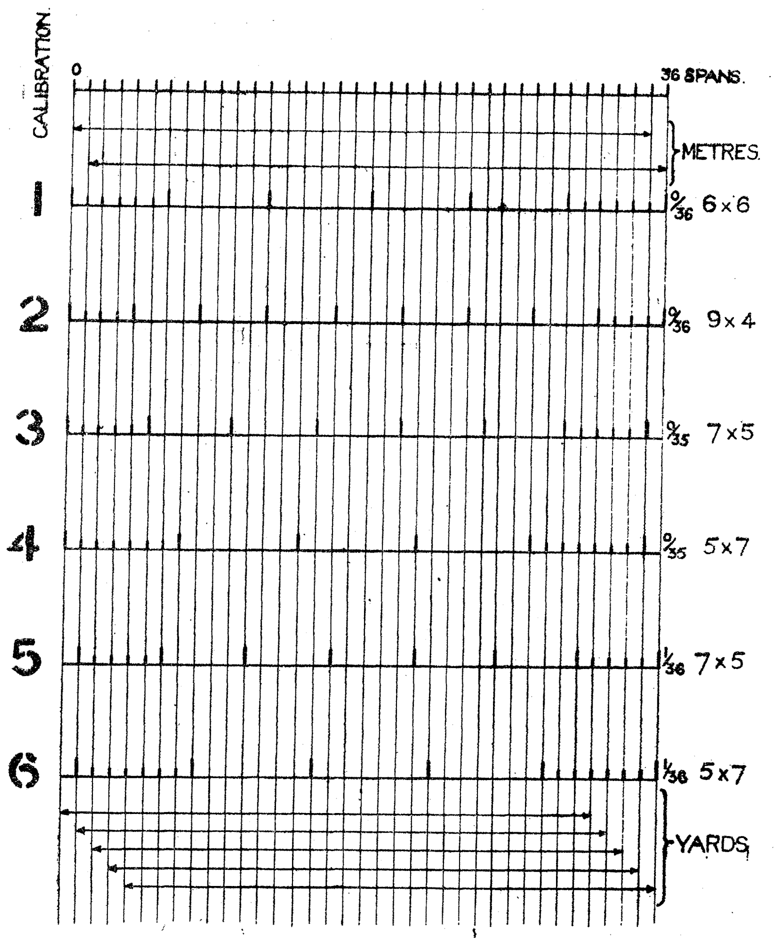

FIG. 1.

The finally determined values of the intervals from zero to the important points at each end of the scale are set out in Table I below in terms of the interval 0/36 assumed correct.

TABLE I.

Interval (spans)	0/1	0/2	0/3	0/4	0/5	0/32	0/33	0/34	0/35	0/36
Error (microns)	−0·269	+3·017	+3·860	+3·621	+5·194	+2·720	+2·450	+1·191	+1·847	0

From these figures may be deduced the various values of e required for the completion of equations (1).

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Let 0 M 35 and 1 M 36 represent the two main metre lengths, and similar symbols, snch as 2 Y 34, the main yard intervals. Then

$$0\,\mathrm{M}\,35 = \frac{35}{36} \times (0/36) + 1 \cdot 847\ \mu,$$

$$1\,\mathrm{M}\,36 = \frac{35}{36} \times (0/36) - (-0 \cdot 269\ \mu) = \frac{35}{36} \times (0/36) + 0 \cdot 269\ \mu,$$

$$\text{Mean} = \frac{0\,\mathrm{M}\,35 + 1\,\mathrm{M}\,36}{2} = \frac{35}{36} \times (0/36) + 1 \cdot 058\ \mu. \quad \ldots \ldots \quad (2)$$

In the same way

$$2\,\mathrm{Y}\,34 = \frac{32}{36} \times (0/36) + 1 \cdot 191\ \mu - 3 \cdot 017\ \mu = \frac{32}{36} \times (0/36) - 1 \cdot 826\ \mu. \quad \ldots \quad (3)$$

Similar expressions are obtainable for the other yard lengths, but, contrary to original intention, only the above main yard interval, for reasons which appear later, enters into the calculations of the required ratio.

From equations (2) and (3), eliminating the (0/36) term, there follows

$$2\,\mathrm{Y}\,34 = \frac{32}{35} \times \frac{0\,\mathrm{M}\,35 + 1\,\mathrm{M}\,36}{2} - 1 \cdot 826\ \mu - \frac{32}{35} \times 1 \cdot 058\ \mu$$

or

$$2\,\mathrm{Y}\,34 = \frac{32}{35} \times \frac{0\,\mathrm{M}\,35 + 1\,\mathrm{M}\,36}{2} - 2 \cdot 79\ \mu. \quad \ldots \ldots \quad (4)$$

Equation (4), which will be recognised as a particular case of equation (1), with $e = -2 \cdot 79\ \mu$, is the one actually employed in the final determination of the ratio of the yard to the metre.

Incidentally the difference 0 M 35 — 1 M 36 = $1 \cdot 85\ \mu - 0 \cdot 27\ \mu = 1 \cdot 58\ \mu$ should be noted for subsequent reference.

Determination of the Yard Intervals.

It was fortunate at this stage of the work that an exceptional opportunity occurred for obtaining a direct comparison of Ni 184 with the Imperial Standard Yard, thus rendering unnecessary all reference of the bar to intermediary copies of the primary standard. These copies, however well attested they may be, are less to be preferred and less likely to give the supremest confidence in the final result.

The decennial comparison of the Imperial Standard Yard and its Parliamentary Copies became due in 1922, and J. E. Sears, in his capacity as Deputy Warden of the Standards, made arrangements for Ni 184 to be included in the contemplated observations. The scheme of comparisons eventually decided on consisted of the complete intercomparison of eight bars, viz., the Imperial Standard Yard, five Parliamentary Copies Nos. 2 to 6, the Board of Trade platinum-iridium bar No. 16 and the N.P.L. bar Ni 184. The magnitude of this closed set of observations led to the decision to standardise only one main yard interval on Ni 184, the interval selected being the symmetrically disposed one, 2/34 spans or 2/38 inches. The inclusion of one or two other yard intervals would have added enormously to the work, and in view of the main object of the comparisons this was not justified. Alternatively, two or three yard intervals could have been compared in a more limited number of observations against the primary standard, but it was thought preferable to concentrate on one interval only and obtain a determination of its length to the highest possible accuracy. It will have been realised that the calibration of the spans gives the relative lengths of the five main yard intervals of Ni 184 to a high degree of accuracy, and that therefore one of them having been well determined, the lengths of the others readily and accurately follow, should they be required for any subsequent purpose.

The observational work on the bars was carried out at the Standards Department of the Board of Trade by means of the Tutton comparator, which is housed in a basement room of the Department. This apparatus is not provided with the usual water bath in which the bars may be immersed, and observations must therefore be made with the standards "in air," but in order to ensure that the temperature of observation should be close to 62° F., the whole room is subjected to thermostatic control.

The comparator was thoroughly overhauled and several improvements made in details. In particular an improved copper sheathed and lagged cover for the traversing table supporting the bars was provided, and proved of great value in maintaining temperatures during the actual progress of observation. The micrometer screws of the microscopes were calibrated throughout against certain $\frac{1}{200}$-inch intervals on Ni 184, which had been specially calibrated in its turn at the N.P.L. The four Tonnelot thermometers employed were also thoroughly re-calibrated by the Thermometry Department of the N.P.L.

Each pair of bars was observed in four different relative positions arranged so that any particular bar was observed an equal number of times on the back and front positions of the traversing table, and so that each line was observed an equal number of times in each miscroscope. Four different observers, F. S. READ* and E. LACEY* of the Standards Department, and L. O. C. JOHNSON and W. H. JOHNSON of the N.P.L., were employed in taking observations, the last named being responsible for the organisation and supervision of the work and subsequently the necessary computations. One at each microscope, these observers worked in pairs, which were varied as much as possible between different sets of observations. Also the two observers changed places half-way through each set of readings, thus counteracting the effect of asymmetrical setting on the lines, which is prevalent to a varying extent in most observers.

Whenever a pair of bars was set up in position at least five hours were allowed to elapse before observations were made, so that the bars and the surrounding apparatus should settle down to a quite steady temperature, which was as close as possible to 62° F., the room meanwhile being locked up and undisturbed. When observations were made, these were carried out as promptly and rapidly as possible after entering the room, the operation taking on the average about 15 minutes.

* Both unhappily since deceased.

Turning now to the actual results obtained, it should first be noted that the arrangements for controlling the temperature worked well, the mean temperature for the whole series being about 16·8° C. Taking the mean temperatures for each pair of bars, these ranged only from 16·76 to 17·15° C., a variation of less than half a degree, and all but three mean temperatures were below 17·0° C. These slight variations from the normal temperature meant that any small uncertainty regarding the values of the coefficients of expansion of the various bars would have a negligible effect on the results when the observed values for the differences between the various pairs of bars were reduced so as to be correct at 16·667° C. (62° F.).

After applying the thermometer and microscope corrections and after making temperature reductions where necessary so that all results were expressed at 62° F., the best values for the differences in length between the various bars were then calculated from the observed results (each being the mean of four) by the method of least squares. Of the resulting residuals (observed value — calculated value) only two appreciably exceed 2 parts in 10,000,000 on the yard length. The two exceptional residuals are of the order of 5 parts in 10,000,000, and both are related to one of the bronze standards which has a particularly poor and indefinite line at one end. The mean residual, neglecting signs, amounts to about $1\frac{1}{2}$ parts in 10,000,000. The figures indicate a high order of accuracy, the probable error of any adjusted value, calculated by the usual mathematical formulæ, being about 2 parts in 30,000,000.

The intercomparison, therefore, has resulted in a well-determined value for the central yard length of Ni 184, as follows:—

$$\text{Ni 184 (2/38 inches)} = 1 \text{ yard} - 0\cdot001339 \text{ inches at } 62^\circ \text{ F.}$$

A report giving a detailed account of the intercomparisons leading to this result will be presented to Parliament by the Standards Department of the Board of Trade.

Determination of the Metre Intervals.

In this section of the work it was realised that direct comparison of the intervals of Ni 184 with the International Prototype Metre would not be possible, but the following means of standardisation were available and were all employed:—

(*a*) Comparison with direct copies of the International Metre at the Bureau International des Poids et Mesures, Sèvres.

(*b*) Comparison with the British National Copy platinum-iridium metre standard No. 16, belonging to the Board of Trade, which has itself been very carefully compared with the copies mentioned in (*a*).

(*c*) Comparison with the N.P.L. Nickel Metre No. 16, which has also been periodically compared with the copies mentioned in (*a*).

The standardisation of the metre intervals of Ni 184 by these three means proved to be a less straightforward task than was at first anticipated. An initial determination with Ni 16 as basis was made at the N.P.L., followed in due course by a determination at the B.I.P.M., Sèvres. Unexpectedly the two results were not in agreement, the divergence being nearly one part in a million. There appeared to be no evidence providing an explanation of the difference, and consequently further work had to be undertaken with a view to finding if possible the root of this trouble, and obtaining new results in better accord. This involved much repetition of observations in order to establish results, and further complications were introduced, first by Ni 16, which in the early stages gave one discordant result out of a number of very concordant ones, and later manifested curious and unexplained changes in length, and secondly by changes in the basic values accepted for the B.I.P.M. standards.

A description of this work, showing how the divergence was eventually reduced to small but not entirely negligible proportions is given below, the facts being recorded for convenience in chronological order.

Table II.—The Lengths of the Metre Intervals of Ni 184.

Date.	Length at 62° F.						Difference. 0 M 35 — 1 M 36.	Value of basis. 1 M +.	No. of bars in closed set.
	0 M 35 = 1 M +			1 M 36 = 1 M +					
	Based on N.P.L. Standard Ni 16.	Based on B. of T. Standard P.I. 16.	Based on B.I.P.M. Standards 26 & T 3.	Based on N.P.L. Standard Ni 16.	Based on B. of T. Standard P.I. 16.	Based on B.I.P.M. Standards 26 & T 3.			
	μ	μ	μ	μ	μ	μ	μ	μ	
(*a*) Feb., 1919 …	+90·20			+88·57			1·63	—22·7	—
(*b*) May, 1919 …	+90·52							—22·7	5
(*c*) Jan., 1920 …			+89·6			+87·8	1·8		—
(*d*) April, 1920 …	+90·48							—22·7	5
(*e*) Oct., 1920 …	+90·24							—22·7	6
(*f*) Feb., 1921 …	+89·62			+88·08			1·54	—22·7	4
(*g*) June, 1921 …	+90·21							—22·7	5
(*h*) Nov., 1921 …				+88·74				—22·7	6
(*i*) June, 1922 …		**+90·40**			**+88·83**		1·57	—0·6	4
(*j*) Nov., 1922 …	+90·20			+88·65			1·55	—22·7	7
(*k*) Nov., 1922 …		**+90·30**			**+88·70**		1·60	—0·6	7
(*l*) June, 1923 …			**+89·93**			**+88·33**	1·60		5
(*m*) Aug., 1923 …	+90·35			+88·66			1·69	—22·2	4
(*n*) Oct., 1923 …		**+90·32**			**+88·69**		1·63	— 0·6	5
(*o*) Feb., 1924 …	+90·36			+88·73			1·63	—22·2	5
(*p*) June, 1924 …			**+89·95**	Maudet		**+88·37**	1·58		5
			+90·20	Johnson		**+88·60**	1·60		5
Mean values from June, 1921 …	+90·28	+90·34	+90·03	+88·70	+88·74	+88·43	1·61		—

The various values obtained for the metre interval of Ni 184 have been summarised for purposes of reference in Table II. Except in two instances (which do not figure in the final computations) these values result from closed sets of observations on four or more bars and their accuracy should consequently be of a high order (say, 2 parts in 10,000,000) comparable with the result obtained for the yard interval. The observational work, except where indicated below, has been carried out by W. H. JOHNSON with the assistance of L. O. C. JOHNSON of the N.P.L.

As already mentioned, the work commenced with Nickel 16 as a basis, one determination being made directly against Ni 16, and a more accurate one in a closed set of five bars at a later date ((*a*) and (*b*) Table II).

Ni 184 was then despatched to the B.I.P.M., Sèvres, where the standardisation was carried out by comparing each metre interval of the bar directly with the Bureau standard No. 26 only. The results ((*c*) Table II), as already noted, were considerably at variance with those obtained at the N.P.L., the difference originally amounting to nearly 1 μ. This difference was subsequently reduced slightly by applying to the observations the true value of the thermal coefficient of expansion of Ni 184 in place of a value that had been assumed at Sèvres. The figures given in Table II are subsequent to this adjustment. The divergence between the two results was too great to be attributable to observational errors, and the cause of the difference had to be sought in some other direction, but there appeared to be nothing to account for it. True, the observations at the B.I.P.M. were not made in a closed set, and, moreover, owing to the unusual dimensions of Ni 184, they had to be carried out with the aid of a comparator other than that usually employed for such work; but these conditions, while possibly contributing some risk of increased error, could only in the worst case account for a portion of the discrepancy.

Two further determinations ((*d*) and (*e*) Table II), made at the N.P.L. with Ni 16 as basis, served to confirm very closely the values previously obtained. So far the N.P.L. results had been satisfactorily consistent, but in February, 1921, comparisons made in a closed set of four bars with Ni 16 again as basis, resulted in values ((*f*) Table II) that were very close to the Sèvres values of 1920. This result differed so much (about $0 \cdot 6\ \mu$) from the values previously obtained at the Laboratory that but for its agreement with the Sèvres values its accuracy would have been immediately placed under suspicion, and in spite of the agreement with Sèvres this suspicion still could not be avoided when it was found that another bar included in the observations, No. 139, had also apparently changed by $0 \cdot 6\ \mu$. The facts appeared to point to some change having actually taken place in the length of Ni 16, though it was difficult to affirm that this was really the case. The result in any case was deemed to be unreliable, an opinion confirmed, as will be seen, by subsequent events. It remained throughout the rest of the work, an isolated result differing from all the others and apparently inexplicable.

Further closed sets of observations with Ni 16 as basis made in the succeeding months served to give results ((*g*) and (*h*) Table II) that confirmed the original values obtained for Ni 184.

At this stage of the work an opportunity for independent verification of the metre lengths of Ni 184 presented itself in the chance of a comparison with the British National copy of the Metre, P.I. 16. Side by side with the comparison of the Imperial Yard and its copies already noted, it was thought advisable by the Standards Department of the Board of Trade to send P.I. 16 to Sèvres for verification, but in order to keep a check on any possible change that might occur in its length during its absence, it was decided to have comparisons of its length made against other bars, both prior to its despatch to Sèvres, and on its return. Both these comparisons were made at the N.P.L. and included in the closed sets carried out were the two metre lengths of Ni 184. It was found by this means that P.I. 16 suffered no change during the journey to and from Sèvres, and, further, its re-verified length formed a new basis for the lengths of the metre intervals of Ni 184. The resulting values of the latter ((*i*) and (*k*) Table II) were found to agree closely with the previous N.P.L. values, with the exception of that of February, 1921.

A still further determination against Ni 16 in the meantime ((*j*) Table II), again confirmed the N.P.L. values (always excepting the divergent one).

In view of the generally consistent repetition of the N.P.L. determinations, the decision was made to send Ni 184, together with Ni 16, again to Sèvres, with the request that, in view of the important ultimate object of the determinations, the measurement of the lengths required should be carried out in a closed set of metre bars. On the previous occasion, as already noted, comparisons were made with only one Bureau bar, and, moreover, owing to the extra overall length of Ni 184 as compared with the Bureau Standards, the observations had been made in a comparator different from that usually employed.

A modification of the usual comparator, however, enabled the Bureau in June, 1923, to carry out the comparisons in their usual way and under conditions closely resembling those at the N.P.L. The conditions, however, differed in one particular, the points of supports for Ni 184 in the Bureau comparator differed slightly from those normally used at the N.P.L. Comment on this is reserved for the present.

Five lengths in all were completely intercompared, the two platinum-iridium Bureau standards, 26 and T3, whose lengths were the basis of the subsequent calculations, Nickel 16, and the two metre intervals of Ni 184. The scheme of comparisons closely resembled that usually followed at the N.P.L., or that followed at the Board of Trade during the intercomparison of the yard bars. The chief difference consisted in the way in which errors arising from the personal asymmetry of the observer in setting on the lines were avoided by taking readings first from the front of the comparator and then an equal number from the rear, the Bureau comparator being specially well adapted for this course. The resulting values ((*l*) Table II) for Ni 184 were accepted with much more confidence than were those of the previous occasion, when the comparison was not carried out in a closed set, but they still differed from the mean of the N.P.L. results, though the original difference between the two Laboratories was now reduced to about 0·4 μ. The divergence certainly could be considered small for most purposes, but it was hoped to obtain for the purpose in view agreement at least within $0 \cdot 2\ \mu$, an amount which represents the maximum observational variation likely to arise in successive determinations of any particular bar.

At this stage it is not proposed to try to find an explanation for these differences, since this is dealt with later on when more information became available, but to turn for a moment to the Bureau report on the length of Nickel 16. This bar had up to this time been considered a particularly stable standard, the various values of its lengths determined periodically by the Bureau not varying by more than $0 \cdot 2\ \mu$ (see Table V later), but the Bureau Report now showed that the bar had suffered an increase in length of $0 \cdot 5\ \mu$, an altogether unexpected and disconcerting piece of news. A comparison of Ni 16 in a closed set, with other bars ((*m*) Table II), immediately on its return from Sèvres apparently confirmed the change, but it was thought desirable to obtain independent evidence on the point, and a comparison against the Board of Trade standard P.I. 16 was decided upon. This was done in a closed set of five bars ((*n*) Table II), the direct observations against P.I. 16 being made on the Tutton comparator at the Standards Department, the remainder on the N.P.L. comparator. The change in length of Ni 16 on this basis was completely confirmed, and the new value for the bar therefore accepted, but with reservation, for having once changed in length, there remained the possibility at any time of a further change.

As a result of this alteration in length some doubt was naturally cast on some of the later N.P.L. determinations, but an exhaustive analysis of all the more recent comparisons, including those made at Sèvres, led to the conclusion that it was highly probable that the change occurred after the bar left the Laboratory, possibly the result of an accidental jar on the journey.

The position at this stage of the work was briefly as follows: The comparisons against Ni 16, allowing for changes in the latter, were, with the one exception noted, quite consistent. The comparisons against P.I. 16 also were in good agreement, and closely confirmed those against Ni 16. The mean of these results differed, however, from the Bureau values by about $0 \cdot 4\ \mu$, and this discrepancy not only gave rise to much anxiety, but, in whatever light it was viewed, defied definite explanation. Owing to the large number of determinations taken against Ni 16 and P.I. 16 and the good agreement obtained amongst them, there was an inclination to place considerable reliance on these results, but at the same time there existed no doubt as to the accuracy of the Bureau determinations. The point remained, therefore, to find an explanation of the divergence between the two, and the only way of doing this appeared to be for an observer from the N.P.L. to visit the Bureau and take observations side by side with a Bureau observer, both working under exactly the same conditions. It would then be possible to find whether the discrepancy was due to the conditions under which the comparisons were made, or whether it would be attributed to some differences or pecularities in observers.

In June, 1924, W. H. Johnson visited the Bureau with this express purpose, taking with him both Ni 184 and Ni 16 for re-verification. It was arranged that the scheme of comparisons carried out at the Bureau in 1923 should be repeated in every detail (see p. 290), and that Monsieur L. MAUDET of the Bureau and W. H. JOHNSON should take complete but independent observations on the five metre lengths involved. In order that conditions should be as closely as possible the same for each observer, and also so that the bars should suffer the minimum amount of handling, it was decided that, for a given relative position of any two bars, each observer should make successively his two sets of observations fore and aft of the comparator.

TABLE III.

Bars compared.	W. H. JOHNSON.			L. MAUDET.		
	Observed.	Calculated.	Obs.-Calc.	Observed.	Calculated.	Obs.-Calc.
	μ	μ	μ	μ	μ	μ
184 (0/35) — 184 (1/36)	+ 1·58	+ 1·61	−0·03	+ 1·77	+ 1·58	+0·19
184 (0/35) — 26	−55·10	−55·26	+0·16	−55·49	−55·56	+0·07
184 (1/36) — 26	−56·92	−56·87	−0·05	−57·08	−57·14	+0·06
184 (0/35) — T3	−56·12	−55·91	−0·21	−56·40	−56·12	−0·28
184 (1/36) — T3	−57·61	−57·52	−0·09	−57·80	−57·70	−0·10
184 (0/35) — 16	−97·42	−97·49	+0·07	−97·61	−97·65	+0·04
184 (1/36) — 16	−99·00	−99·10	+0·10	−99·00	−99·23	+0·23
16 — 26	+42·13	+42·23	−0·10	+42·10	+42·09	+0·01
16 — T3	+41·87	+41·58	+0·29	+41·78	+41·53	+0·25
26 — T3	— 0·65	— 0·65	0·00	— 0·44	— 0·56	+0·12
		Mean residual.	0·11		Mean residual.	0·13

Each pair of bars was observed in four different relative positions, arranged so that each line on a bar was observed equally often through each microscope. Taking observations to the rear of the comparator is equivalent to the addition of the other four possible relative positions of the bar, if these be considered as relative to the observer and not to the comparator, the eight sets of observations thus corresponding to the eight possible different relative positions of the two bars.

The work was carried out in the Brunner comparator, as in 1923, and the time of the year selected for it ensured that the temperature should be close to 62° F., that is, the temperature at or near which all other observations had been made, not only those in determining the yard and metre lengths of Ni 184, but also those in connection with the re-verification of the various basic standards (Ni 16, P.I. 16 and the Bureau bars 26 and T 3).

The number of observations was double that which is usually considered necessary and sufficient to ensure a reliable determination, and the resulting values could therefore be accepted with the greatest confidence. The results obtained by each observer for each pair of bars are set out separately in Table III, which shows the observed values for each difference, the calculated values, and the residuals.

A glance at the residuals shows that the observations have been made to a high accuracy. The mean residual for Johnson is 0·11 μ and for Maudet 0·13 μ, and it follows that the final determinations can be relied on to within 0·1 μ. The resulting values for Ni 184 are given in Table II (*p*), and it will be seen that each observer has substantially confirmed the values previously obtained by him, still leaving an appreciable discrepancy between the two. The two values approach each other slightly by amounts that might easily be attributable to experimental error, and thus apparently reduce the divergence to about 0·3 μ. This difference is solely related to the N.P.L. bar Ni 184, for in all results where this bar does not figure, there is closer agreement, as, for example, in the values obtained for Ni 16 where the difference is less than 0·1 μ.

Incidentally, it may be mentioned that the results in Table III illustrate the order of accuracy attained throughout the whole of the determinations of the same kind referred to in the foregoing pages.*

Although the conditions under which the observations at Sèvres had been made were as closely as possible like those prevailing at the N.P.L., there remained one variation the effect of which requires to be considered. This is concerned with the points of support of Ni 184. At the N.P.L. the same two points of support, definitely marked on the bar, were invariably used when the latter was under observation. At the B.I.P.M. these points could not be used, owing to certain difficulties that prevented a redisposition of the supporting rollers in the Brunner comparator, and other points were used instead. Theoretically the effect of this slight variation in conditions should be negligible, but a practical test was also made in order to find whether it had any real effect on the length of the bar. A supplementary comparison of Ni 184 and Ni 16 was therefore made in the Brunner comparator, using temporary supports arranged as at the N.P.L. There was some difficulty in arranging this, and the device adopted was not one that could be employed in a long series of observations. Both observers took complete sets of observations on the two bars set up in this way, exactly similar to those made in the main series of observations, with the result that each observer repeated very closely the value he obtained in the main comparison, as Table IV below clearly shows.

Table IV.

Supports as at	Observed values of 184 0/35 — 16	
	W. H. Johnson.	L. Maudet.
	μ	μ
N.P.L.	−97·40	−97·68
B.I.P.M.	−97·42	−97·61

* *Added November* 6, 1927.—Attention may also be called at this point to the high degree of uniformity exhibited in Table II among the differences between the values obtained for the two metre intervals 0 M 35 and 1 M 36 on Ni 184. The mean of these differences, 1·61 μ, may be compared with the value 1·58 μ directly determined, to a high order of accuracy, in the course of the calibration of the $1\frac{1}{8}$-inch spans (see p. 285).

There is apparently no definite explanation to be found for the persistent difference between the N.P.L. and B.I.P.M. values for Ni 184. Since W. H. JOHNSON has substantially repeated at the Bureau his N.P.L. results, it can be concluded that the conditions under which the observations have been made do not account for the divergence. Since the two institutions, so far as line standards are concerned, carry out their work on such closely similar lines, there appears to be no reason why a result obtained by one laboratory should not be repeated by the other. In the present case the precautions taken in procedure should eliminate any effect on the final results arising from personal equation in the ordinary sense, and this is confirmed by the fact that for a considerable portion of the work, namely, that in which Ni 184 does not figure, there is reasonable agreement. The divergence is throughout associated with Ni 184, and the only explanation that presents itself is that possibly some peculiarity exists in the graduations of this bar whereby the two observers interpret one or both of the lines of Ni 184 somewhat differently. This opinion, however, can be regarded only as a surmise arrived at by a close scrutiny of results and by a process of exhaustion of probable causes. The discrepancy appears to be one that cannot be either eliminated or definitely accounted for, and its existence appears to point pretty definitely to a limitation of the accuracy which can ever be expected in dealing with comparisons of this kind.

Turning now to the values obtained by W. H. JOHNSON and L. MAUDET for the length of Ni 16, it has already been seen that the two observers have obtained closely agreeing results. Taking the mean of the two as a very probable value for its length, the result unfortunately shows a further change in Ni 16, indicating that the bar has this time shortened, returning almost to its original value (see Table V). This was not unexpected, but no less difficult to explain. The question of possible damage to the bar does not this time hold as an explanation, as W. H. JOHNSON had the bar in his personal care throughout the journey to and fro. Unfortunately, time did not permit of a comparison of Ni 16 with other bars immediately before it was taken to Sèvres, and so it cannot be said whether this change had taken place at the N.P.L. previous to the journey, but a comparision at the N.P.L. subsequent to its return did not, as expected, uphold the new value at Sèvres, but confirmed the previous shortening.

These various changes are certainly bewildering, and it would be interesting to be able to find the reason for them. Due consideration of everything that could possibly afford an explanation of these changes, and the clues are but slight, would make it appear improbable that any real change has occurred. It seems more likely that certain conditions exist, which are accidently reproducible on occasion, and which lead to apparent changes in the length of the bar. Such changes might, for example, be due to some peculiarity in the formation of the graduation lines, whereby slight differences in the nature of the illumination used caused them to take on alternative appearances. The discordant N.P.L. value of February, 1921, would also be accounted for by a similar temporary change.

Table V below shows the various lengths of Ni 16 determined from time to time by the Bureau.

TABLE V.

Date of B.I.P.M. determinations.	Length of Ni 16 at 0° = 1 M +
	μ
January, 1904	—22·8
December, 1908	—22·6
October, 1912	—22·6
March, 1920	—22·7
July, 1923	—22·2
June, 1924	—22·6

The Bureau Standards.

During the progress of this investigation, the B.I.P.M. found it necessary to adjust the basic values of their standards, including T3 and 26, by an amount approximating to $+ 0\cdot4\ \mu$, this alteration being retrospective. It should be stated here that in the foregoing pages this change has throughout been taken into account, and where necessary values adjusted accordingly.

The Ratio of the Yard to the Metre.

It is now possible to turn to the actual calculation of the ratio of the yard to the metre. Of the three main divisions of the work in connection with Ni 184, (*a*) the calibration of the spans, (*b*) the determination of the length of the central yard interval, and (*c*) the determinations of the lengths of the main metre intervals, the first two presented no difficulty, and the results obtained can be accepted with the greatest confidence. The determination of the metre intervals, as has already been seen, presented considerable difficulty, particularly in the attempt to obtain agreement between the N.P.L. and the B.I.P.M. values. Moreover, the changes that have occurred in the length of Ni 16 have added complications, and results based on this bar do not therefore give the same confidence as those based on other bars, as, say, P.I. 16.

It is probable that no further observations on Ni 184 would reduce the discrepancy which has been the source of so much trouble and extra effort, and that the figures obtained by the two observers must be regarded as equally well determined.

It remains, therefore, to consider the most satisfactory way of dealing with the various results obtained (Table II), so as to arrive at the best possible value of the required ratio. Since some of these can be accepted with more confidence than others, it has been decided for the purpose in view to discard a portion of them, having due regard of course to the various uncertainties affecting them, thus leaving only the best determined and most reliable values. At the outset it has been thought best to omit all those prior to June, 1921, thus definitely excluding the original determination of the B.I.P.M. (January, 1920), and the discordant result of February, 1921, based on Ni 16. Of the remainder, the values based on Ni 16 are, owing to the variations in lengths which this bar has exhibited, less to be preferred than those based on P.I. 16 or on 26 and T3, despite their very close agreement (see mean values at foot of Table II) with those based on P.I. 16. But owing to the slight uncertainty associated with them, it has been deemed advisable to exclude these also, though their omission does not affect the final results appreciably. There are thus retained for the purpose of the final calculations the values (shown in heavy type) based on P.I. 16 and on 26 and T3.

In deciding the appropriate weight to attach to these results, it must be borne in mind that the comparisons of Ni 184 with the Imperial Standard Yard were made by W. H. Johnson and other observers, all of whose readings agreed with his, so that in this stage of the work it is his interpretation of the lines on Ni 184 which has been accepted.

For the immediate purpose of obtaining a true comparison of the yard with the metre, Johnson's interpretation of the lines on Ni 184 is consequently to be preferred to Maudet's. On the other hand it would not be right to exclude Maudet's values altogether. Taking the direct arithmetical mean of the comparisons shown in heavy type has the effect of giving double weight to Johnson's interpretation, and this manner of weighting the values has been accepted as affording on the whole the simplest and most satisfactory compromise between several possible alternatives.

For convenience, the various basic data required for the calculation of the ratio are given below. First, equation (4) is reproduced.

(i) *Calibration.*

$$2\ \text{Y}\ 34 = \frac{32}{35} \times \frac{0\ \text{M}\ 35 + 1\ \text{M}\ 36}{2} - 2\cdot79\ \mu. \quad \ldots \ldots \quad (4)$$

(ii) *Yard Interval.*

$$2\ \text{Y}\ 34 = 1\ \text{yard} - 0\cdot001339\ \text{inches at}\ 62^\circ\ \text{F}.$$
$$= 35\cdot998661\ \text{inches at}\ 62^\circ\ \text{F}.$$

(iii) *Metre Intervals.*

Table VI.

Basis.	Observer.	Length of interval at 62° F.	
		0 M 35 = 1 M. +	1 M 36 = 1 M. +
		μ	μ
P.I. 16	Johnson . . .	+90·40	+88·83
		+90·30	+88·70
		+90·32	+88·69
26 and T3	Johnson . . .	+90·20	+88·60
	Maudet . . .	+89·93	+88·33
		+89·95	+88·37
Means		+90·183	+88·587

From these figures we get

$$\frac{0\text{ M }35 + 1\text{ M }36}{2} = 1 \cdot 00008938 \text{ metres},$$

so that

$$35 \cdot 998661 \text{ inches} = 2 \text{ Y } 34$$
$$= \frac{32}{35} \times 1 \cdot 00008938 - 0 \cdot 00000279 \text{ metres}$$
$$= 0 \cdot 91436743 - 0 \cdot 00000279 \text{ metres}$$
$$= 0 \cdot 91436464 \text{ metres},$$

and hence

$$1 \text{ metre} = \frac{35 \cdot 998661}{0 \cdot 91436464} = 39 \cdot 370137 \text{ inches.}\dagger$$

The present legal conversion factor, obtained at the Bureau International in 1895, is

$$1 \text{ metre} = 39 \cdot 370113 \text{ inches},$$

from which the result now found differs by only about 6 parts in 10 millions, an amount which it is satisfactory to observe is probably no more than the possible combined experimental errors of the present and of the former determinations.

[*Added, March* 16, 1928.—Since the foregoing was accepted for publication, it has been suggested to the authors by Dr. C. E. GUILLAUME, Director of the Bureau International, that use should be made of the results of certain investigations which have been in progress for a considerable period at the Bureau, leading to revised values for the coefficients of thermal expansion of various platinum-iridium standard metre bars (including those of the Bureau),* although these investigations have not yet been fully completed, and the revised values have consequently not yet been formally adopted by the International Committee. If these new results are taken into account, the values at 62° F. of the several metre bars involved in the present comparisons have to be amended as follows :—

TABLE VII.

Standard.	Length of standard = 1 M.+				New — old at 16·667° C.
	at 0° C.		at 16·667° C.		
	Old.	New.	Old.	New.	
	μ	μ	μ	μ	μ
26	+1·15	+1·55	+145·55	+145·33	−0·22
T_3	+1·85	+1·77	+146·03	+145·78	−0·25
P.I. 16	+0·60	+0·60	+145·10	+144·78	−0·32

* See 'La Création du Bureau International des Poids et Mesures et Son Œuvre,' Ch. Ed. GUILLAUME, pp. 113 *et seq.* (1927).

† For final result see next page.

In view of these changes it then becomes necessary to revise Table VI above, which thus becomes :—

TABLE VIII.

Basis.	Observer.	Length of interval at 62° F.	
		0 M 35 = 1 M +.	1 M 36 = 1 M +.
		μ	μ
P.I. 16	Johnson	+90·08	+88·51
		+89·98	+88·38
		+90·00	+88·37
26 & T_3	Johnson	+89·97	+88·36
	Maudet	+89·72	+88·14
		+89·70	+88·10
	Means	+89·908	+88·310

These amended values, as compared with Table VI, represent a mean change of approximately 1 part in 4,000,000. At the same time the self-consistency of Johnson's results, as also the concordance between the mean values obtained from his results and from Maudet's, are slightly improved, although the unexplained difference between the two observers still persists.

M. Guillaume, in signifying his concurrence with these amendments, writes, "Toutefois, il faut bien spécifier qu'en ce qui concerne les valeurs absolues, nous avons encore des expériences à faire, pour pouvoir donner une valeur définitive de la formule de dilatation." It is not anticipated, however, that any further change in the thermal coefficients is likely to be of a magnitude which would have material influence on the results given in this paper.

Re-calculating the value of the ratio of the Yard to the Metre from the revised figures, we get

$$\frac{0\text{ M }35 + 1\text{ M }36}{2} = 1\cdot00008911 \text{ metres,}$$

so that

$$35\cdot998661 \text{ inches} = \frac{32}{35} \times 1\cdot00008911 - 0\cdot00000279 \text{ metres}$$

$$= 0\cdot91436440 \text{ metres,}$$

whence

$$\mathbf{1\ metre} = \frac{35\cdot998661}{0\cdot91436440} = \mathbf{39\cdot370147\ inches}$$

and

$$\mathbf{1\ yard} = 36 \times \frac{0\cdot91436440}{35\cdot998661} = \mathbf{0\cdot91439841\ metres}$$

or

$$\mathbf{1\ inch = 25\cdot399956\ millimetres,}$$

as compared with the present legal conversion factors

$$1 \text{ metre} = 39\cdot370113 \text{ inches}$$

or

$$1 \text{ inch} = 25\cdot399978 \text{ millimetres.}$$

It will be noticed that the difference between the new value of the ratio and the current legal value based on the determinations of CHANEY and BENOIT in 1895 is somewhat increased by this final revision. Part, at any rate, of this increase would probably disappear again if the results of the earlier comparisons were re-calculated making due allowance for the retrospective effect of the new determinations of the coefficients of thermal expansion of the platinum-iridium bars. There are, however, other factors which render such a re-calculation at the present date of little value or interest, and it has not been considered worth while to pursue it. In any case the difference between the old and new values is still less than 1 part in a million—*i.e.*, it is still no more than might be accounted for by the possible combined errors of all the operations entering into the two independent determinations. The new determination, therefore, does not appear to suggest evidence of any actual change in the lengths of the primary standards of either the Yard or Metre during the last 30 years.]

APPENDIX.

THE CALIBRATION OF THE SPANS OF NI 184.

Section I.—Notes on Calibration of Scales.

A. *Notation.*

1. m, n, p, q, etc., are integral numbers representing the *theoretical distances* of certain graduations from the zero of the scale in appropriate units. They are, in fact, the labels of these graduations.

2. m/n means the *actual distance* between the graduations marked m and n. For convenience m is taken as the smaller number of the pair.

3. $_p(m/n)_q$ means the distance m/n in terms of the distance p/q as correct, *i.e.*,

$$_p(m/n)_q = m/n - \frac{n-m}{q-p}\cdot p/q.$$

This quantity will naturally be expressed in much smaller units than m/n or p/q.

Thus, whereas the latter distances might be expressed in centimetres, $_p(m/n)_q$ will generally be expressed in microns (μ).

B. *Procedure.*

Suppose a scale is divided into x theoretically equal parts. The usual method of calibration,* in the case where x is not a very large number (say $x < 12$), leads to a square schedule containing x^2 smaller squares, each of the latter containing an experimental value for a general term $(m-1)/m - (n-1)/n$. Where x is large, this involves too many observations, and it is usual to make a preliminary calibration into, say, a divisions, each of the latter containing b smaller divisions, so that $ab = x$.

The principal graduations are then 0, b, $2b, \ldots ab$ and the first square schedule contains a^2 smaller squares each with experimental value for a term

$$[(\alpha - 1)\, b/\alpha b - (\beta - 1)\, b/\beta b],$$

where α and β are any integers between or including 1 and a. This is referred to as Part I of the calibration.

Part II makes a further subdivision of two or more of the groups of b units, and consists of one or more squares containing b^2 experimental values for a general term

$$[(m-1)/m - (n-1)/n],$$

where all the "m" terms now refer to units confined to one group of b and all the "n" terms to those confined to another group of b.

The summation of each column in the square of Part I gives the terms

$$a\,[(\alpha - 1)\, b/\alpha b - \frac{1}{a} \,.\, 0/x],$$

dividing by a gives the term within the brackets which is by definition

$${}_0[(\alpha - 1)\, b/\alpha b]_x.$$

The summation of each column in Part II gives the terms

$$b\left[(m-1)/m - \frac{1}{b} \,.\, (\beta - 1)\, b/\beta b)\right],$$

dividing by b gives the term within the brackets which is by definition

$${}_{(\beta-1)\,b}[(m-1)/m]_{\beta b}.$$

Whenever the expression $a \times b$ is used of a calibration in the following sections, it is to be taken to indicate that Part I of the calibration consisted of a determination of the a chief divisions of the length concerned, and that the b minor divisions comprising the first and last of the main divisions respectively were individually compared. Thus, for example, 0/35 (7 × 5) indicates the calibration of the length 0 to 35 primarily into 7 main divisions and secondly into the first and last 5 smaller units.

* See, for example, GLAZEBROOK, 'Dict. of App. Phys.,' vol. 3, "Line Standards."

C. *Some Simple Transformations.*

Being given the value of a certain length on a bar in terms of another length on the same bar as correct, it is often desirable to express it in terms of some different length as correct, or the latter in terms of the former as correct, *e.g.* :—

1. Given ${}_p(m/n)_q$, find ${}_m(p/q)_n$—

By def.: $${}_p(m/n)_q = m/n - \frac{n-m}{q-p} \cdot p/q,$$

therefore
$$p/q = \frac{q-p}{n-m}\,[m/n - {}_p(m/n)_q],$$

therefore
$${}_m(p/q)_n = p/q - \frac{q-p}{n-m} \cdot m/n = -\frac{q-p}{n-m} \cdot {}_p(m/n)_q.$$

2. Given ${}_p(m/n)_q$, find ${}_r(m/n)_s$—

By def.: $${}_r(m/n)_s = m/n - \frac{n-m}{s-r} \cdot r/s$$
$$= m/n - \frac{n-m}{s-r}\left[\frac{s-r}{q-p} \cdot p/q + {}_p(r/s)_q\right]$$
$$= m/n - \frac{n-m}{s-r}\left[\frac{s-r}{q-p} \cdot p/q - \frac{s-r}{q-p} \cdot {}_r(p/q)_s\right]$$
$$= m/n - \frac{n-m}{q-p} \cdot p/q + \frac{n-m}{q-p} \cdot {}_r(p/q)_s$$
$$= {}_p(m/n)_q + \frac{n-m}{q-p} \cdot {}_r(p/q)_s.$$

These two propositions are practically self-evident on inspection, but when performing many such transformations it is useful to have the formulæ tabulated.

D. *Graphical Explanation.*

The propositions under heading C can be illustrated graphically.

Suppose a bar is divided into x parts all nominally equal. Let the integers 0, 1, 2 . . . x be abscissæ and plot as ordinates the difference between the various distances 0 to n and their ideal values $\frac{n}{x}$ times $0/x$, *i.e.*, they are errors in position of the various graduations relative to $0/x$ as correct, or the quantities ${}_0(0/n)_x$ of the notation in paragraph A.

In fig. 2 such a graph for an imaginary bar calibrated into x equal parts is shown by the full circles. The end points of the graph are necessarily zero, for ${}_0(0/0)_x$ and ${}_0(0/x)_x$ are by definition equal to zero. This does not imply that in making readings on the two end graduations no errors of observation have been made, and in actual fact, by forcing

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the two end points to zero, we compel the other points to take on themselves errors $\frac{n-x}{x}\varepsilon_0 - \frac{n}{x}\varepsilon_x$ in addition to the error proper to the observation on each point n, where ε_0 and ε_x are the errors of observation on the zero and x line respectively.

Referring to the definition of ${}_0(m/n)_x$, it will be seen from the graph that the vertical height between the points P and Q represents ${}_0(p/q)_x$, *i.e.*, the distance p/q in terms of $0/x$ as correct, this quantity being taken as positive when Q is higher than P.

To find the error of position of any point in relation to p/q as correct, we first observe that if, according to this particular calibration, the interval p/q is, say, short, in relation to $0/x$ as correct, by an amount QQ′, then the length of any other interval m/n must be longer, in relation to p/q as correct, by an amount $\frac{n-m}{q-p} \times$ QQ′ than it is in relation to $0/x$ as correct.

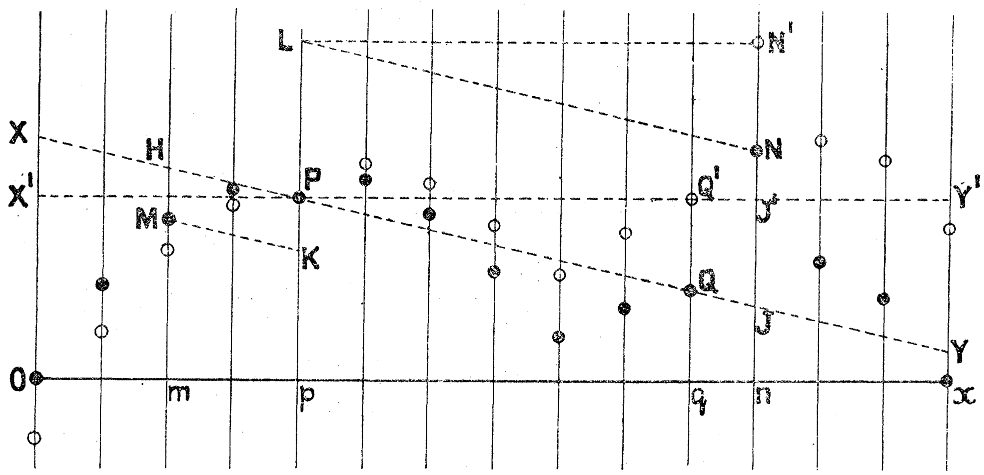

Fig. 2.

Consider the interval p/n. The error of p/n in relation to $0/x$ as correct is NJ′. By similar triangles PQQ′, PJJ′, we get $\frac{n-p}{q-p} \times$ QQ′ $=$ JJ′. The total error of the interval p/n (or the error of position of the line n) in relation to p/q as correct is therefore equal to NJ (or LP), the vertical distance from N to the line XY joining P and Q.

Similarly, the error of the interval m/p (or the error of position of the line m) in relation to p/q as correct is — MH (or PK) and the error of the whole interval m/n in relation to p/q as correct is

$$\text{NJ} + \text{HM} = \text{LP} + \text{PK} = \text{LK},$$

which is the vertical distance between lines drawn through M and N, respectively, parallel to the line PQ.

This is a simple geometrical conception, but not particularly convenient for purposes of computation. We can readily transfer to rectangular co-ordinates as follows: With the horizontal line X′PY′ as base replot the points, as shown by the plain circles, moving each vertically through a distance equal to the corresponding intercept between the lines XY and X′Y′. We then have N′J′ = NJ, and similarly for other points, so that the errors of position of the various points, according to this calibration, in relation to p/q as correct, are now measured in the ordinary rectangular sense from the new base line X′Y′.

The operation just described is as if each of the points P Q M N, etc., were attached by short rigid rods of fixed lengths, constrained to slide in vertical guides, with their ends resting on a straight rod X Y, which is then tilted into the position X′Y′, which makes PQ′ horizontal. And it may be noted that this imaginary mechanism will continue to represent the results of the calibration in question, no matter into what position the rod is tilted, or to what extent it may be moved bodily in a vertical direction. The value of any interval, in terms of any other as correct, will always be found by measuring the vertical distance between lines drawn from the terminal points of the one interval, parallel to the line joining the terminal points of the other interval.

E. *Combination of Several Independent Calibrations.*

When we come to combine the results of two or more independent calibrations, not necessarily each involving the same array of points, the first solution which suggests itself is to take any common interval, reduce all the graphs to the basis of this interval as correct, and take the means of all the observed results for each of the other points, on this basis. This simple procedure, however, is only correct if every calibration involves the same array of points, for the terminal points of the chosen interval will themselves be associated with certain observational errors in each of the calibrations, and these errors will react on the calculated errors of the other points in a manner which depends in part on the particular terminal points chosen, so that if different common intervals be chosen, different results will be obtained.

If, however, we retain the conception of the mechanically linked graphs, each of which, as has been shown, may be either tilted, or lifted bodily, in relation to the others, without impairing its own validity as a record of observations, there is one, and only one, ideal arrangement in which the whole of the graphs can be combined together to give the best possible mean results. This arrangement is conditioned by the fact that in the adjusted positions of all the graphs the sum of the squares of all the differences taken for all points, between the values of any point given by any component graph and that given by the resulting mean graph, shall be a minimum.

In the computation the vertical movement and the tilt are provided for by the introduction of two unknowns A and B for each independent calibration. To keep the magnitude of the quantities in the calculation reasonably small, the diagram of each calibration is imagined first to be laid down in some position known to be fairly near to the correct one, and then to be moved up and down, and tilted, so that each point has its final position changed by an amount $A + nB$. The object of the calculation is then to determine the "A's" and "B's" so as to arrive at the best mean values for the combination of the whole series of calibrations.

Let x = whole number of units dealt with in all. Thus 0 and x are the outermost points of the scale calibrated.

n = theoretical distance from the *zero* of the scale to a certain point on the scale. n is an integral number and is the label of a certain graduation.

C_n = number of separate calibrations involving the point n.

p_N = number of points observed in calibration N.

A_N = vertical movement necessary on the graph of calibration N.

B_N = tilt factor necessary on the graph of calibration N.

Thus any point n is moved vertically through a distance $A_N + nB_N$ by these two movements.

n_0 = desired ideal value of $0/n$ in terms of the whole as correct.

n_N = $0/n$ in terms of the whole as correct according to the Nth calibration, *i.e.*, ${}_0(0/n)_x$ from the Nth calibration according to the unabbreviated notation of paragraph A.*

$\sum\limits_{C_n} f$ = sum of all the functions f, whatever they be, of the point n extended over the C_n calibrations involving the point n.

$\sum\limits_{p_N} f$ = sum of all the p_N functions f, whatever they be, of the p_N points involved in calibration N.

For example—

$\sum\limits_{C_n} A_N = A_1 + A_2 + \ldots + A_N + \ldots$ to C_n terms assuming that the point n has been included in all the C_n calibrations 1, 2 ... N, etc., mentioned above.

$\sum\limits_{p_N} A_N = A_N + A_N + \ldots + A_N + \ldots = p_N A_N$, p_N being the number of points observed in the Nth calibration.

$\sum\limits_{p_N} \frac{n}{C_N} = \frac{0}{C_0} + \frac{1}{C_1} + \frac{2}{C_2} + \ldots + \frac{n}{C_n} + \ldots$ including p_N terms corresponding to the p_N points observed in the Nth calibration.

We have now to determine the values of the "A's" and "B's" for the optimum arrangement.

When the graphs have been arranged in conjunction, the observational equations from any particular calibration, say, the Nth, are p_N in number and are as follows :—

$$n_N + A_N + nB_N - n_0 = 0,$$

$$m_N + A_N + mB_N - m_0 = 0, \text{ etc., etc.}$$

* *Note.*—It may happen that the Nth calibration does not directly involve either one or both of the points 0 and x. In that case the meaning of n_N is to be understood to imply the value of $0/n$ in terms of $0/x$ resulting from the Nth calibration, when this calibration has been laid down in an arbitrary position (ultimately to be the best position) in relation to the original datum points for 0 and x, against which the whole system of graphs is plotted.

These equations express the fact that the provisional value n_N for the point n in this calibration has theoretically been brought into coincidence with the unknown ideal value n_0 by the vertical movement A_N and the tilt B_N of the whole of the graph to which the value n_N belongs.

In these equations as applied to one calibration only there are $p_N + 2$ unknowns, namely $0_0, 1_0, 2_0 \ldots n_0 \ldots$ (p_N of these) and the two quantities A_N and B_N. Thus there are two more unknowns than equations. This is in accordance with the fact that the graph possesses two degrees of freedom whilst still expressing the same relative positions of the points.

The introduction of the equations resulting from another calibration, however (say the Mth), in general only adds few or no unknowns of the type n_0, m_0 plus the two unknowns A_M and B_M, while the total number of equations is now $p_N + p_M$. With a number of different calibrations the total number of equations is likely to be greatly in excess of the total number of unknowns.

Normal Equations are now formed as follows :—

Consider the general equation

$$A_N + nB_N - n_0 = -n_N$$

obtained from the Nth calibration and concerning the point n in that calibration. There are C_n such equations containing the term n (see notation).

Multiplying each by the coefficient of n_0 and summing, we have

$$\sum_{C_n} A_N + n \sum_{C_n} B_N - C_n n_0 = -\sum_{C_n} n_N. \quad \ldots \quad (1)$$

For every point $0, 1, 2 \ldots n \ldots$ etc., which has been observed at all there is a normal equation of type (1).

Returning to the general equation, there are p_N equations containing the term A_N, all of them belonging to the Nth calibration. Summing them to obtain the normal equation we have

$$p_N A_N + B_N \sum_{p_N} n - \sum_{p_N} n_0 = -\sum_{p_N} n_N. \quad \ldots \quad (2)$$

Similarly, multiplying each of the p_N equations by the coefficient of B_N and summing, we have

$$A_N \sum_{p_N} n + B_N \sum_{p_N} n^2 - \sum_{p_N} n n_0 = -\sum_{p_N} n n_N. \quad \ldots \quad (3)$$

For each calibration there is a pair of such normal equations as (2) and (3).

These normal equations can theoretically be solved for

$$A_1,\ A_2 \ldots A_N,\ B_1,\ B_2 \ldots B_N,\ 0_0,\ 1_0,\ 2_0, \ldots n_0,$$

but in practice a determinant containing, say, 30 unknowns of the type n_0 and 6 each of type A_N and B_N (a case actually occurring in the comparison of the yard and the metre desired above) is intractable.

It may be simplified as follows :—

From equation (1)

$$n_0 = \frac{1}{C_n}\left(\sum_{C_n} A_N + n \sum_{C_n} B_N + \sum_{C_n} n_N\right), \quad \ldots \ldots \ldots \quad (1)$$

substituting in (2) and (3) :—

$$p_N A_N + B_N \sum_{p_N} n - \sum_{p_N}\left(\frac{1}{C_n}\left\{\sum_{C_n} A_N + n \sum_{C_n} B_N + \sum_{C_n} n_N\right\}\right) = -\sum_{p_N} n_N \quad \ldots \quad (4)$$

$$A_N \sum_{p_N} n + B_N \sum_{p_N} n^2 - \sum_{p_N}\left(\frac{n}{C_n}\left\{\sum_{C_n} A_N + n \sum_{C_n} B_N + \sum_{C_n} n_N\right\}\right) = -\sum_{p_N} n n_N. \quad \ldots \quad (5)$$

The term $\sum_{p_N}\left(\frac{1}{C_n}\sum_{C_n} A_N\right)$ may be arranged as follows :—

$$\begin{aligned}\sum_{p_N}\left(\frac{1}{C_n}\sum_{C_n} A_N\right) = & \frac{1}{C_0}(A_1 + A_2 + \ldots + A_N + \ldots), \quad C_0 \text{ terms.} \\ & + \frac{1}{C_1}(A_1 + A_2 + \ldots + A_Q + \ldots), \quad C_1 \text{ terms.} \\ & + \ldots \\ & + \ldots \\ & + \frac{1}{C_n}(A_1 + A_2 + \ldots + A_R + \ldots), \quad Cn \text{ terms.} \\ & + \ldots,\end{aligned}$$

the terms included being suitably selected in accordance with the points observed in the various calibrations. Supposing the Nth calibration does not include a certain point, say m, the term $\frac{1}{C_m}(A_1 + A_2 + \ldots)$ will be missing from the above expansion. Further, the term $\frac{1}{C_p}(A_1 + A_2 + \ldots)$ contains only the terms $A_1, A_2, \ldots, A_N$, etc., belonging to those calibrations 1, 2 ... N, etc., which have included the point p. Hence, the condition for the inclusion of any term $\frac{1}{C_p} A_M$ in the double summation $\sum_{p_N}\left(\frac{1}{C_n}\sum_{C_n} A_N\right)$ is that the point p should be common to the Mth and Nth calibrations.

Using the symbol $\sum_{p_{MN}}$ to represent a sum of functions of points common to the Mth and Nth calibrations, we may therefore write

$$\begin{aligned}\sum_{p_N}\frac{1}{C_n}\sum_{C_n} A_N &= \sum A_M \sum_{p_{MN}} \frac{1}{C_n} \\ &= A_1 \sum_{p_{1N}} \frac{1}{C_n} + A_2 \sum_{p_{2N}} \frac{1}{C_n} + \ldots + A_M \sum_{p_{MN}} \frac{1}{C_n} + \ldots + A_N \sum_{p_N} \frac{1}{C_n} {}^{*} + \ldots.\end{aligned}$$

* Notice that $\sum_{p_{NN}} f = \sum_{p_N} f$.

TABLE IX.

A_1	A_2	......	A_M	......	A_N	......	B_1	B_2	......	B_M	......	B_N	......	
$-\sum_{p_{1N}} \frac{1}{C_n}$	$-\sum_{p_{2N}} \frac{1}{C_n}$	......	$-\sum_{p_{MN}} \frac{1}{C_n}$	......	$-\sum_{p_N} \left(\frac{1}{C_n} - 1\right)$	......	$-\sum_{p_{1N}} \frac{n}{C_n}$	$-\sum_{p_{2N}} \frac{n}{C_n}$	......	$-\sum_{p_{MN}} \frac{n}{C_n}$	......	$-\sum_{p_N} \left(\frac{n}{C_n} - n\right)$	......	$\sum_{p_N} \left(\frac{1}{C_n} \sum \frac{n_N}{c_n} - n_N\right)$
$-\sum_{p_{1N}} \frac{n}{C_n}$	$-\sum_{p_{2N}} \frac{n}{C_n}$	......	$-\sum_{p_{MN}} \frac{n}{C_n}$	......	$-\sum_{p_N} \left(\frac{n}{C_n} - n\right)$	......	$-\sum_{p_{1N}} \frac{n^2}{C_n}$	$-\sum_{p_{2N}} \frac{n^2}{C_n}$	......	$-\sum_{p_{MN}} \frac{n^2}{C_n}$	......	$-\sum_{p_N} \left(\frac{n^2}{C_n} - n^2\right)$	......	$\sum_{p_N} \left(\frac{n}{C_n} \sum \frac{n_N}{c_n} - n n_N\right)$

Similarly with any other term $\sum\limits_{p_N} \frac{1}{C_n} \sum\limits_{C_n} f_N$.

The factors in the normal equations derived from the Nth calibration have now been reduced by elimination to the form shown in Table IX.

The quantity $\frac{1}{C_n} \sum\limits_{C_n} n_N$ is the mean value of the point n given by the calibrations which include it. It may therefore be written $\bar{n}$. Similarly $\frac{n}{C_n} \sum\limits_{C_n} n_N = n\bar{n}$. The final determinant for the evaluation of A_1, A_2, etc., and B_1, B_2, etc., is therefore as shown in Table X.

This determinant appears to be soluble, for there are twice as many unknowns as there are calibrations and also twice as many equations as there are calibrations.

But it has been shown that any determination of the relative position of the points on the bar can be represented by a graph which still possesses two degrees of freedom. Hence, the above determinant must possess two redundant equations, and, in fact, if the coefficients of any term such as A_N or B_N be summed either in the α equations or the β equations, it will be found that all the sums are equal to zero. In other words, one of the α equations and one of the β equations are not independent.

For example, take the coefficients of A_N in the α equations. Their sum is

$$\sum_{p_{1N}} \frac{1}{C_n} + \sum_{p_{2N}} \frac{1}{C_n} + \ldots + \sum_{p_{MN}} \frac{1}{C_n} + \ldots + \sum_{p_N} \left(\frac{1}{C_n} - 1\right) + \ldots.$$

Expand each term thus:—

$$\sum_{p_N} \left(\frac{1}{C_n} - 1\right) = \sum_{p_N} \frac{1}{C_n} - p_N = \left(\frac{1}{C_0} + \frac{1}{C_1} + \ldots + \frac{1}{C_n} + \ldots\right) - p_N,$$

$$\sum_{p_{1N}} \frac{1}{C_n} = \left(\frac{1}{C_0} + \frac{1}{C_1} + \ldots + \frac{1}{C_m} + \ldots\right),$$

$$\sum_{p_{2N}} \frac{1}{C_n} = \left(\frac{1}{C_0} + \frac{1}{C_1} + \ldots + \frac{1}{C_q} + \ldots\right),$$

the terms included in each line being suitably selected in accordance with the points observed in the various calibrations.

Now consider a vertical column of terms, say $\frac{1}{C_1}$. By the definition of C_1, it is the number of calibrations which have involved the point 1. Having found the term $\frac{1}{C_1}$ among the p_N terms belonging to the Nth calibration (*i.e.*, the top horizontal row) the number of times $\frac{1}{C_1}$ occurs in the vertical column must be $= C_1$, so that the sum of the column must $= 1$. Similarly for all other vertical columns of terms $\frac{1}{C_n}$. Hence, the sum of all the terms within brackets is equal to p_N, the number of terms within the top bracket, and the sum of the coefficients of A_N in the equations is zero.

Similarly the sum of the coefficients of any other term, whether A or B, can be shown to be zero both in the α equations and the β equations. The redundancy of two of the equations in the determinant is therefore proved.

TABLE X.—Final Determinant.

Calibration	Equation	A_1	A_2	----	A_M	----	A_N	----	B_1	B_2	----	B_M	----	B_N	----	
1	α	$\sum_{p_1}\left(\frac{1}{C_n}-1\right)$	$\sum_{p_{12}}\frac{1}{C_n}$	----	$\sum_{p_{1M}}\frac{1}{C_n}$	----	$\sum_{p_{1N}}\frac{1}{C_n}$	----	$\sum_{p_1} n\left(\frac{1}{C_n}-1\right)$	$\sum_{p_{12}}\frac{n}{C_n}$	----	$\sum_{p_{1M}}\frac{n}{C_n}$	----	$\sum_{p_{1N}}\frac{n}{C_n}$	----	$\sum_{p_1}(n_1-\bar{n})$
	β	$\sum_{p_1} n\left(\frac{1}{C_n}-1\right)$	$\sum_{p_{12}}\frac{n}{C_n}$	----	$\sum_{p_{1M}}\frac{n}{C_n}$	----	$\sum_{p_{1N}}\frac{n}{C_n}$	----	$\sum_{p_1} n^2\left(\frac{1}{C_n}-1\right)$	$\sum_{p_{12}}\frac{n^2}{C_n}$	----	$\sum_{p_{1M}}\frac{n^2}{C_n}$	----	$\sum_{p_{1N}}\frac{n^2}{C_n}$	----	$\sum_{p_1} n(n_2-\bar{n})$
2	α	$\sum_{p_{12}}\frac{1}{C_n}$	$\sum_{p_2}\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_{2M}}\frac{1}{C_n}$	----	$\sum_{p_{2N}}\frac{1}{C_n}$	----	$\sum_{p_{12}}\frac{n}{C_n}$	$\sum n\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_{2M}}\frac{n}{C_n}$	----	$\sum_{p_{2N}}\frac{n}{C_n}$	----	$\sum_{p_2}(n-\bar{n})$
	β	$\sum_{p_{12}}\frac{n}{C_n}$	$\sum_{p_{2M}} n\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_{2M}}\frac{n}{C_n}$	----	$\sum_{p_{2N}}\frac{n}{C_n}$	----	$\sum_{p_{12}}\frac{n^2}{C_n}$	$\sum_{p_2} n^2\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_{2M}}\frac{n^2}{C_n}$	----	$\sum_{p_{2N}}\frac{n^2}{C_n}$	----	$\sum_{p_2} n(n_2-\bar{n})$
		⋮	⋮		⋮		⋮		⋮	⋮		⋮		⋮		⋮
N	α	$\sum_{p_{1N}}\frac{1}{C_n}$	$\sum_{p_{2N}}\frac{1}{C_n}$	----	$\sum_{p_{MN}}\frac{1}{C_n}$	----	$\sum_{p_N}\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_{1N}}\frac{n}{C_n}$	$\sum_{p_{2N}}\frac{n}{C_n}$	----	$\sum_{p_{MN}}\frac{n}{C_n}$	----	$\sum_{p_N} n\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_N}(n_N-\bar{n})$
	β	$\sum_{p_{1N}}\frac{n}{C_n}$	$\sum_{p_{2N}}\frac{n}{C_n}$	----	$\sum_{p_{MN}}\frac{n}{C_n}$	----	$\sum_{p_N} n\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_{1N}}\frac{n^2}{C_n}$	$\sum_{p_{2N}}\frac{n^2}{C_n}$	----	$\sum_{p_{MN}}\frac{n^2}{C_n}$	----	$\sum_{p_N} n^2\left(\frac{1}{C_n}-1\right)$	----	$\sum_{p_N} n(n_N-\bar{n})$
		⋮	⋮		⋮		⋮		⋮	⋮		⋮		⋮		

The two additional equations necessary to complete the determinant are furnished by the limiting conditions that the errors of the extreme points of the scale must be equal to zero, since the remaining points have their errors expressed in terms of these as correct.

Thus equation (1) states

$$n_0 = \frac{1}{C_n} \sum_{C_n} (n_N + A_N + nB_N)$$

if 0 and x are the extremities of the portion calibrated, we may write

$$0_0 = 0 = \frac{1}{C_0} \sum_{C_0} (0_N + A_N), \quad \ldots \ldots \ldots \ldots \quad (6)$$

$$x_0 = 0 = \frac{1}{C_x} \sum_{C_x} (x_N + A_N + xB_N). \quad \ldots \ldots \ldots \quad (7)$$

All the terms 0_1 0_2 . . . x_1 x_2 . . . will in general have been put equal to zero to start with, thus reducing these equations to

$$\sum_{C_0} A_N = 0,$$

$$\sum_{C_x} A_N + x \sum_{C_x} B_N = 0,$$

respectively.

The solution of the determinant must now be carried out by any method which appears feasible. Since the number of unknowns is twice the number of calibrations, it might appear that the solution becomes more and more tedious as the number of calibrations increases. This is not the case, because the coefficients of A_N and B_N in the equations derived from the Nth calibration $\left(e.g., \sum_{p_N}\left(\frac{1}{C_n} - 1\right) \text{ and } \sum_{p_N} n\left(\frac{1}{C_n} - 1\right) \text{ from equation } N\alpha\right)$ become larger in proportion to the other coefficients as the total number of calibrations is increased. Thus methods of successive approximation become more feasible and give a result which more quickly approaches the desired degree of approximation.

Having evaluated all the quantities A_1, A_2 . . . B_1, B_2, etc., the desired values 1_0, 2_0 . . . n_0, etc., are obtained by substitution in equation (1).

It is interesting to compare the quantities

$$\begin{array}{ll} n_1 + A_1 + nB_1 & \\ n_2 + A_2 + nB_2 & \qquad \text{etc.} \\ \dots\dots\dots\dots & \\ \dots\dots\dots\dots & \end{array}$$

individually with the final calculated value n_0. The differences are true residuals (observed — calculated), since the quantities $(n + A + nB)$, etc., are true observed values. The calculation has merely shown the best position, defined by the parameters A and B, in which to superpose the component graphs.

Section II.—Calculation of Final Results.

Taking the six independent calibrations involved in the present comparison of the yard with the metre, the preparation and solution of the determinant for evaluating the six quantities A_1, A_2 . . . A_6 and the six quantities B_1, B_2 . . . B_6 is the main part of the calculation involved.

Section I E should be referred to in connection with the following tables.

Table XI, giving the preparation of the coefficients in the determinant, has in horizontal rows the three quantities 1, n, n^2 for all points included in each calibration. The horizontal summation of the rows gives the quantities p_N, $\sum_{p_N} n$, $\sum_{p_N} n^2$ for each calibration.

These are required in the terms $\sum_{p_N}\left(\frac{1}{C_n} - 1\right)$, $\sum_{p_N} n\left(\frac{1}{C_n} - 1\right)$ and $\sum_{p_N} n^2\left(\frac{1}{C_n} - 1\right)$ respectively.

Table XI.—Normal Equations—Preparation of Coefficients.

	$n \rightarrow$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1						1						1
	n	0	1	2	3	4	5	6						12						18
	n^2	0	1	4	9	16	25	36						144						324
2	1	1	1	1	1	1				1				1				1		
	n	0	1	2	3	4				8				12				16		
	n^2	0	1	4	9	16				64				144				256		
3	1	1	1	1	1	1	1					1					1			
	n	0	1	2	3	4	5					10					15			
	n^2	0	1	4	9	16	25					100					225			
4	1	1	1	1	1	1	1	1	1							1				
	n	0	1	2	3	4	5	6	7							14				
	n^2	0	1	4	9	16	25	36	49							196				
5	1		1	1	1	1	1	1					1					1		
	n		1	2	3	4	5	6					11					16		
	n^2		1	4	9	16	25	36					121					256		
6	1		1	1	1	1	1	1	1	1							1			
	n		1	2	3	4	5	6	7	8							15			
	n^2		1	4	9	16	25	36	49	64							225			
	C_n	4	6	6	6	6	5	4	2	2		1	1	2		1	2	2		1
	$\frac{1}{C_n}$	0·25	0·17	0·17	0·17	0·17	0·20	0·25	0·50	0·50		1·00	1·00	0·50		1·00	0·50	0·50		1·00
	$\frac{n}{C_n}$	0	0·17	0·33	0·50	0·67	1·00	1·50	3·50	4·00		10·00	11·00	6·00		14·00	7·50	8·00		18·00
	$\frac{n^2}{C_n}$	0	0·17	0·67	1·50	2·67	5·00	9·00	24·50	32·00		100·00	121·00	72·00		196·00	112·50	128·00		324·00

	$n \rightarrow$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36				
1	1						1						1	1	1	1	1	1	1	→	p_1	=	17
	n						24						30	31	32	33	34	35	36	→	$\sum_{p_1} n$	=	306
	n^2						576						900	961	1024	1089	1156	1225	1296	→	$\sum_{p_1} n^2$	=	8786
2	1		1				1				1				1	1	1	1	1	→	p_2	=	16
	n		20				24				28				32	33	34	35	36	→	$\sum_{p_2} n$	=	288
	n^2		400				576				784				1024	1089	1156	1225	1296	→	$\sum_{p_2} n^2$	=	8044
3	1		1					1					1	1	1	1	1	1		→	p_3	=	16
	n		20					25					30	31	32	33	34	35		→	$\sum_{p_3} n$	=	280
	n^2		400					625					900	961	1024	1089	1156	1225		→	$\sum_{p_3} n^2$	=	7760
4	1			1							4	1	1	1	1	1	1	1		→	p_4	=	18
	n			21							28	29	30	31	32	33	34	35		→	$\sum_{p_4} n$	=	315
	n^2			441							784	841	900	961	1024	1089	1156	1225		→	$\sum_{p_4} n^2$	=	8757
5	1			1					1					1	1	1	1	1	1	→	p_5	=	16
	n			21					26					31	32	33	34	35	36	→	$\sum_{p_5} n$	=	296
	n^2			441					676					961	1024	1089	1156	1225	1296	→	$\sum_{p_5} n^2$	=	8336
6	1				1							1	1	1	1	1	1	1	1	→	p_6	=	18
	n				22							29	30	31	32	33	34	35	36	→	$\sum_{p_6} n$	=	333
	n^2				484							841	900	961	1024	1089	1156	1225	1296	→	$\sum_{p_6} n^2$	=	9405
	C_n		2	2	1		2	1	1		2	2	4	5	6	6	6	6	4				
	$\frac{1}{C_n}$		0·50	0·50	1·00		0·50	1·00	1·00		0·50	0·50	0·25	0·20	0·17	0·17	0·17	0·17	0·25				
	$\frac{n}{C_n}$		10·00	10·50	22·00		12·00	25·00	26·00		14·00	14·50	7·50	6·20	5·33	5·50	5·67	5·83	9·00				
	$\frac{n^2}{C_n}$		200·00	220·50	484·00		288·00	625·00	676·00		392·00	420·50	225·00	192·20	170·67	181·50	192·67	204·17	324·00				

$\sum_{p_{MN}} \frac{1}{C_n}$

M↓ N→	1	2	3	4	5	6
1	4·76	2·86	2·26	2·51	2·26	2·51
2		4·86	2·11	2·11	2·11	2·11
3			5·26	2·26	1·76	2·51
4				5·51	2·51	3·26
5					5·26	2·26
6						5·51

$\sum_{p_{MN}} \frac{n}{C_n}$

M↓ N→	1	2	3	4	5	6
1	85·2	51·0	38·7	40·2	41·7	49·2
2		87·0	34·0	38·0	41·0	37·0
3			91·2	38·7	31·2	46·2
4				96·7	43·2	58·2
5					97·2	41·7
6						100·7

$\sum_{p_{MN}} \frac{n^2}{C_n}$

M↓ N→	1	2	3	4	5	6
1	2193	1438	1176	1185	1284	1509
2		2190	954	1146	1206	1110
3			2214	1176	951	1289
4				2439	1181	1630
5					2430	1284
6						2583

TABLE XII.—Normal Equations—Preparation of Numerical Values.

	n →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	
1	n_1	0	-0·32	+2·92	+3·92	+3·63	+5·32	+4·40						+1·72						+4·22						+4·72						+0·16	+0·05	+2·58	+2·34	+1·10	+1·73	0	
	$n_1-\bar{n}$	0	0	-0·04	+0·11	+0·06	+0·17	+0·17						+0·09						0						+0·06						+0·07	-0·15	-0·12	-0·09	-0·08	-0·10	0	→ $\Sigma_{p_1}(n_1-\bar{n}) = +0·24^{2}$
	$n_1+A_1+nB_1$																																						
	$n(n_1-\bar{n})$	0	0	-0·08	+0·33	+0·24	+0·85	+1·02						+1·08						0						+1·44						+2·10	-4·65	-3·84	-2·97	-2·72	-3·52	0	→ $\Sigma_{p_1} n(n_1-\bar{n}) = -10·70^{5}$
2	n_2	0	-0·42	+2·64	+3·77	+3·48				+4·20				+1·55				+4·58				+0·04				+4·60				+3·08				+2·78	+2·43	+1·16	+1·90	0	
	$n_2-\bar{n}$	0	-0·10	-0·12	-0·04	-0·09				+0·05				-0·08				-0·06				+0·11				-0·06				+0·04				+0·08	0	-0·02	+0·07	0	→ $\Sigma_{p_2}(n_2-\bar{n}) = -0·22^{4}$
	$n_2+A_2+nB_2$																																						
	$n(n_2-\bar{n})$	0	-0·10	-0·24	-0·12	-0·36				+0·40				-0·96				-0·84				+2·20				-1·44				+1·12				+2·56	0	-0·68	+2·45	0	→ $\Sigma_{p_2} n(n_2-\bar{n}) = +3·99^{6}$
3	n_3	0	-0·30	+2·99	+3·83	+3·63	+5·00					+1·71					+4·48					-0·18					+5·02					+0·08	-0·17	+2·51	+2·28	+1·13	+1·82		
	$n_3-\bar{n}$	0	+0·02	+0·03	+0·02	+0·06	-0·15					0					0					-0·11					0					-0·01	-0·37	-0·19	-0·15	-0·05	-0·01		→ $\Sigma_{p_3}(n_3-\bar{n}) = -0·91^{2}$
	$n_3+A_3+nB_3$																																						
	$n(n_3-\bar{n})$	0	+0·02	+0·06	+0·06	+0·24	-0·75					0					0					-2·20					0					-0·30	-11·47	-6·08	-4·95	-1·70	-0·35		→ $\Sigma_{p_3} n(n_3-\bar{n}) = -27·42^{5}$
4	n_4	0	-0·17	+3·24	+4·00	+3·73	+5·44	+4·30	+5·13							+2·83							+3·37							+3·01	+0·65	+0·28	+0·48	+2·84	+2·57	+1·18	+1·82		
	$n_4-\bar{n}$	0	+0·15	+0·28	+0·19	+0·16	+0·29	+0·07	+0·16							0							+0·03							-0·03	+0·25	+0·19	+0·28	+0·14	+0·14	0	-0·01		→ $\Sigma_{p_4}(n_4-\bar{n}) = +2·29^{8}$
	$n_4+A_4+nB_4$																																						
	$n(n_4-\bar{n})$	0	+0·15	+0·56	+0·57	+0·64	+1·45	+0·42	+1·12							0							+0·63							-0·84	+7·25	+5·70	+8·68	+4·48	+4·62	0	-0·35		→ $\Sigma_{p_4} n(n_4-n) = +35·08^{4}$
5	n_5		-0·37	+2·80	+3·61	+3·34	+4·97	+4·18					+0·62					+4·69					+3·32					+7·03					+0·47	+2·90	+2·65	+1·37	+2·01	0	
	$n_5-\bar{n}$		-0·05	-0·16	-0·20	-0·23	-0·16	-0·05					0					+0·05					-0·02					0					+0·27	+0·20	+0·22	+0·19	+0·18	0	→ $\Sigma_{p_5}(n_5-\bar{n}) = +0·24^{2}$
	$n_5+A_5+nB_5$																																						
	$n(n_5-\bar{n})$		-0·05	-0·32	-0·60	-0·92	-0·80	-0·30					0					+0·80					-0·42					0					+8·37	+6·40	+7·26	+6·46	+6·30	0	→ $\Sigma_{p_5} n(n_5-\bar{n}) = +35·08^{4}$
6	n_6		-0·37	+2·98	+3·71	+3·60	+5·04	+4·04	+4·81	+4·10							+4·47							+2·74							+0·15	-0·16	+0·15	+2·60	+2·33	+1·11	+1·71	0	
	$n_6-\bar{n}$		-0·05	+0·02	-0·10	+0·03	-0·11	-0·19	-0·16	-0·05							-0·01							0							-0·25	-0·25	-0·05	-0·10	-0·10	-0·07	-0·12	0	→ $\Sigma_{p_6}(n_6-\bar{n}) = -1·55^{6}$
	$n_6+A_6+nB_6$																																						
	$n(n_6-\bar{n})$		-0·05	+0·04	-0·30	+0·12	-0·55	-1·14	-1·12	-0·40							-0·15							0							-7·25	-7·50	-1·55	-3·20	-3·30	-2·38	-4·20	0	→ $\Sigma_{p_6} n(n_6-\bar{n}) = -32·95^{7}$
																																							Small raised figures
	$\Sigma_{C_n} n_\kappa$	0	-1·95	+17·77	+22·84	+21·41	+25·77	+16·92	+9·94	+8·30		+1·71	+0·62	+3·27		+2·83	+8·95	+9·27		+4·22		-0·14	+6·69	+2·74		+9·32	+5·02	+7·03		+6·09	+0·80	+0·36	+0·98	+16·21	+14·60	+7·05	+10·99	0	adjust the two
																																							sums to zero
	$\frac{1}{C_n}\Sigma_{C_n} n_\kappa = \bar{n}$	0	-0·32	+2·96	+3·81	+3·57	+5·15	+4·23	+4·97	+4·15		+1·71	+0·62	+1·63		+2·83	+4·48	+4·64		+4·22		-0·07	+3·34	+2·74		+4·66	+5·02	+7·03		+3·04	+0·40	+0·09	+0·20	+2·70	+2·43	+1·18	+1·83	0	

TABLE XIII.—Norma

		A_1	A_2	A_3	A_4	A_5	A_6	B_1	B_2	B_3	B_4	B_5	B_6	
1	α	-12·24	2·86	2·26	2·51	2·26	2·51	-2208	51·0	38·7	40·2	41·7	49·2	+0·22
	β	-220·8	51·0	38·70	40·2	41·7	49·2	-6593	1438	1176	1185	1284	1509	-10·73
2	α	2·86	-11·14	2·11	2·11	2·11	2·11	51·0	201·0	34·0	38·0	41·0	37·0	-0·24
	β	51·0	-201·0	34·0	38·0	41·0	37·0	1438	-5854	954	1146	1206	1110	+3·96
3	α	2·26	2·11	-10·74	2·26	1·76	2·51	38·7	34·0	-188·8	38·7	31·2	46·2	-0·92
	β	38·7	34·0	-188·8	38·7	31·2	46·2	1176	954	-5546	1176	951	1289	+27·45
4	α	2·51	2·11	2·26	-12·49	2·51	3·26	40·2	38·0	38·7	-218·3	43·2	58·2	+2·28
	β	40·2	38·0	38·7	-218·3	43·2	58·2	1185	1146	1176	-6318	1181	1630	+35·04
5	α	2·26	2·11	1·76	2·51	-10·74	2·26	41·7	41·0	31·2	43·2	-198·8	41·7	+0·22
	β	41·7	41·0	31·2	43·2	-198·8	41·7	1284	1206	951	1181	-5906	1284	+32·15
6	α	2·51	2·11	2·51	3·26	2·26	-12·49	49·2	37·0	46·2	58·2	41·7	-232·3	-1·56
	β	49·2	37·0	46·2	58·2	41·7	-232·3	1509	1110	1289	1630	1284	-6822	-32·97

Approximate elimination of B's	A_1	A_2	A_3	A_4	A_5	A_6	B^{100}_1	B^{100}_2	B^{100}_3	B^{100}_4	B^{100}_5	B^{100}_6
→	3197	-760	+635	-167	-570	-569	0	-19	+4	-4	+8	+9
							Inserting values of B's for 2nd approximation to A's →					
→	-649	2481	-552	-471	-411	-491	-9	0	-8	+8	+2	+6
→	-522	-528	2392	-522	-387	-520	+7	-8	0	+7	+7	-13
→	-708	-504	-583	3128	-642	-789	+5	+10	+12	0	-15	-12
→	-506	-432	-420	-624	2390	-506	+9	-2	+5	-20	0	+9

Approximate elimination of A's

A_1	A_2	A_3	A_4	A_5	A_6	B^{100}_1	B^{100}_2	B^{100}_3	B^{100}_4	B^{100}_5	B^{100}_6	
0	+22	+75	+181	-34	-144	+955	-190	-174	-168	-194	-227	+537
	Inserting approximate values of A's →											
+19	0	+131	+17	-95	+35	-168	+823	-110	-149	-151	-145	-269
+32	+97	0	+32	-8	-65	-156	-112	+703	-156	-127	-150	+556
+133	-41	+29	0	+24	-44	-174	-174	-181	+905	-154	-222	+174
+5	-61	+45	+105	0	+5	-164	-143	-119	-122	+711	-164	-896

	B^{100}_1	B^{100}_2	B^{100}_3	B^{100}_4	B^{100}_5	B^{100}_6	
→	+6811	-5655	-297	0	-140	-391	+516
→	-127	+6242	-5244	0	-200	+2	-402
→	-1192	-910	+4306	0	-984	-1207	+2498
→	-1537	-1364	-1175	0	+5658	-1590	-7035

	B^{100}_1	B^{100}_2	B^{100}_3
→	+2296	-2361	-0
Equation of limiting condition	*+152	+152	0
→	-30	+98	0
	*+28	+28	0
→	-159	-137	0
	*+163	+163	0

…nal Equations—Summarised Solution.

			A_1	A_2	A_3	A_4	A_5	A_6		
	-382 -381	→	+6940	-6390	+6	-389	-165			-1119 -1134
	+220 +228	→	-318	+6065	-5725	+45	-93			+463 +477
	-8 -7	→	-107	-376	+5340	-4969	+70			+842 +889
	-676 -696	→	+103	+225	+91	+5198	-5537			-1863 -1880
	+509 +505									

	A_1	A_2	A_3	A_4		
→	+4549	-10898	+6160	-284		-1175 -1200
Equation of limiting condition	* 1	1	1	1		0
→	-494	+5988	+1472	-7051		+1700 +1755
	* 1	1	1	1		0
→	-150	-533	+7650	-7005		+1169 +1208
	* 1	1	1	1		0

	A_1	A_2	A_3		
→	+4835	-10114	+6444		-1175 -1200
→	+6557	+13039	+8523		+1700 +1755
→	+6855	+6472	+14635		+1169 +1208

	A_1	A_2
→	-19	-3100
→	+301	+1088

B_3^{100}	B_4^{100}	B_5^{100}	B_6^{100}		
0 0	-0 0	-43 +132	-132 +132		+1817 -152
0 0	0 0	-27 +28	-28 +28		-19 -32
0 0	0 0	+459 +163	-163 +163		-541 -188

etc etc, whence →

A_2		
-3100		-382 -392·0
+1083		+120 +123·3

1st Approximation

$A_1 = -0·0482$
$A_2 = +0·1235$
$A_3 = +0·0478$
$A_4 = -0·1231$
$A_5 = +0·2257$
$A_6 = +0·1144$

2nd Approximation

$A_1 = -0·0489$
$A_2 = +0·1268$
$A_3 = +0·0494$
$A_4 = -0·1273$
$A_5 = +0·2251$
$A_6 = +0·1206$

$100 B_1 = +0·3737$
$100 B_2 = -0·3865$
$100 B_3 = +0·3417$
$100 B_4 = +0·0774$
$100 B_5 = -1·1570$
$100 B_6 = +0·0158$

[*To face p.* 313.]

The vertical addition of the figures in the rows commencing 1 gives C_n, the number of calibrations involving each point n.

Underneath are placed $\frac{1}{C_n}$, $\frac{n}{C_n}$ and $\frac{n^2}{C_n}$ for evaluating the terms $\sum_{p_{MN}} \frac{1}{C_n}$, etc.

In the latter summation a list of points common to the Mth and Nth calibrations is made and the summation of the quantities $\frac{1}{C_n}$, $\frac{n}{C_n}$ and $\frac{n^2}{C_n}$ over these points is performed. The results are shown in the small squares at the lower part of the table.

Table XII contains the tentative values of n_1, n_2, etc., obtained directly by laying down the observations of the 1st, 2nd, etc., calibrations in the apparently most suitable provisional positions. Next, the differences $n_1 - \bar{n}$, $n_2 - \bar{n}$, etc., between the individual values n_1, n_2, etc., and $\bar{n}$, the mean of them all, are inserted. Underneath are n times these differences, *i.e.*, $n\,(n_1 - \bar{n})$, etc. The horizontal summation of these two rows gives the numerical values to insert on the right-hand side of the determinant.

Passing to the solution of the latter (Table XIII) it will be seen that the equations Nα and Nβ derived from the Nth calibration always have large coefficients for the two unknowns A_N and B_N, while the values of the remainder of the coefficients approximate to fixed quantities for the A's and B's respectively. Advantage is taken of this to obtain a solution by successive approximations. The first step, given on the right of the initial determinant, consists in the elimination of each B_N between each pair of equations Nα and Nβ. On account of the fact that one such pair of equations is redundant (see Section I E), equations 6α and 6β are omitted. Since the B's are only of the order of 1/30 of the A's, the coefficients of B_1, B_2, etc., have all been divided by 100, so that the unknowns are 100 B_1, 100 B_2, etc. In order to see what terms are negligible compared with others, it is only necessary to multiply the coefficients of 100 B_1, etc., by 3 and to compare these figures with the coefficients of A_1, etc.

It is clear that the first step has reduced the equations to a form in which the B terms are quite small in comparison with the A terms. To obtain a first approximation to the A's, the former are neglected *pro tem.* and the equations solved in the straightforward way, introducing at the correct stage the equation of condition,

$$\sum_{C_0} A_N = 0,$$

i.e.,

$$A_1 + A_2 + A_3 + A_4 = 0.$$

This expresses the fact that the 0 end of the final graph is to be brought down to the zero line. [The C_0 calibrations which have involved the point 0 are Nos. 1, 2, 3 and 4.]

Having obtained the six first approximations to $A_1 \ldots A_6$, the next step is to perform the same operation on the A's in the determinant as was previously performed on the B's, *i.e.*, the elimination of A_N in each pair of equations Nα and Nβ, leaving a determinant strong in B terms and weak in A terms (see underneath the main determinant).

TABLE XIV.—Calculation of Final Values.

	n ⟶	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$A_1 = -0.049$	n_1	0	-0.32	+2.92	+3.92	+3.63	+5.32	+4.40						+1.72						+4.22
1	$+A_1$	-0.049	-0.049	-0.049	-0.049	-0.049	-0.049	-0.049						-0.049						-0.049
	$+nB_1$	0	+0.004	+0.007	+0.011	+0.015	+0.019	+0.022						+0.045						+0.069
$B_1 = +0.00374$	$n+A_1+nB_1$	-0.049	-0.365	+2.878	+3.882	+3.596	+5.290	+4.373						+1.716						+4.238
	Residuals	-0.049	-0.096	-0.139	+0.022	-0.025	+0.096	+0.111						+0.043						
$A_2 = +0.127$	n_2	0	-0.42	+2.84	+3.77	+3.48				+4.20				+1.55				+4.58		
2	$+A_2$	+0.127	+0.127	+0.127	+0.127	+0.127				+0.127				+0.127				+0.127		
	$+nB_2$	0	-0.004	-0.008	-0.012	-0.015				-0.031				-0.046				-0.062		
$B_2 = -0.00386$	$n_2+A_2+nB_2$	+0.127	-0.297	+2.959	+3.885	+3.592				+4.296				+1.631				+4.645		
	Residuals	+0.127	-0.028	-0.058	+0.025	0.029				+0.037								-0.043		
$A_3 = +0.049$	n_3	0	-0.30	+2.99	+3.83	+3.63	+5.00					+1.71					+4.48			
3	$+A_3$	+0.049	+0.049	+0.049	+0.049	+0.049	+0.049					+0.049					+0.049			
	$+nB_3$	0	+0.003	+0.007	+0.010	+0.014	+0.017					+0.034					+0.051			
$B_3 = +0.00342$	$n_3+A_3+nB_3$	+0.049	-0.248	+3.046	+3.889	+3.693	+5.066					+1.793					+4.580			
	Residuals	+0.049	+0.021	+0.029	+0.029	+0.072	-0.128										-0.006			
$A_4 = -0.127$	n_4	0	-0.17	+3.24	+4.00	+3.75	+5.44	+4.50	+5.13							+2.85				
4	$+A_4$	-0.127	-0.127	-0.127	-0.127	-0.127	-0.127	-0.127	-0.127							-0.127				
	$+nB_4$	+0	+0.001	+0.002	+0.002	+0.003	+0.004	+0.005	+0.006							+0.011				
$B_4 = +0.00077$	$n_4+A_4+nB_4$	-0.127	-0.296	+3.115	+3.875	+3.606	+5.317	+4.178	+5.008							+2.714				
	Residuals	-0.127	-0.027	+0.098	+0.015	-0.015	+0.125	-0.084	+0.036											
$A_5 = +0.225$	n_5		-0.37	+2.80	+3.61	+3.34	+4.97	+4.18					+0.62					+4.69		
5	$+A_5$		+0.225	+0.225	+0.225	+0.225	+0.225	+0.225					+0.225					+0.225		
	$+nB_5$		-0.012	-0.023	-0.035	-0.046	-0.058	+0.069					-0.127					-0.185		
$B_5 = -0.01157$	$n_5+A_5+nB_5$		-0.157	+3.002	+3.800	+3.519	+5.137	+4.04					+0.718					+4.730		
	Residuals		+0.112	+0.015	-0.060	-0.102	-0.087	+0.074										+0.042		
$A_6 = +0.121$	n_6		-0.37	+2.98	+3.71	+3.60	+5.04	+4.04	+4.81	+4.10							+4.47			
6	$+A_6$		+0.121	+0.121	+0.121	+0.121	+0.121	+0.121	+0.121	+0.121							+0.121			
	$+nB_6$		+0.000	+0.000	+0.000	+0.001	+0.001	+0.001	+0.001	+0.001							+0.002			
$B_6 = +0.00016$	$n_6+A_6+nB_6$		-0.249	+3.101	+3.851	+3.722	+5.162	+4.162	+4.932	+4.222							+4.593			
	Residuals		+0.020	+0.084	-0.029	+0.101	-0.052	-0.100	-0.038	-0.037							+0.006			
	$\sum_{C_n}(n_N+A_N+nB_N)$	0.000	-1.612	+1.401	+23.162	+21.728	+25.972	+17.049	+9.940	+8.518		+1.793	+0.718	+5.347		+2.714	+9.173	+9.375		+4.238
	C_n	4	6	6	6	6	5	4	2	2		1	1	2		1	2	2		1
	$n_0 = \frac{1}{C_n}\sum_{C_n}(n_N+A_N+nB_N)$	0	-0.269	+3.017	+3.860	+3.621	+5.194	+4.262	+4.970	+4.259		+1.793	+0.718	+1.673		+2.714	+4.586	+4.686		+4.238

	n ⟶	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
$A_1 = -0.049$	n_1						+4.72						+0.16	+0.05	+2.58	+2.34	+1.10	+1.73	0
1	$+A_1$						-0.049						-0.049	-0.049	-0.049	-0.049	-0.049	-0.049	-0.049
	$+nB_1$						+0.090						+0.112	+0.116	+0.120	+0.123	+0.127	+0.131	+0.135
$B_1 = +0.00374$	$n+A_1+nB_1$						+4.761						+0.223	+0.117	+2.651	+2.414	+1.178	+1.812	+0.086
	Residuals						+0.063						+0.064	-0.101	-0.069	-0.036	-0.013	-0.035	+0.084
$A_2 = +0.127$	n_2		+0.04				+4.60				+3.08				+2.78	+2.43	+1.16	+1.90	0
2	$+A_2$		+0.127				+0.127				+0.127				+0.127	+0.127	+0.127	+0.127	+0.127
	$+nB_2$		-0.077				-0.093				-0.108				-0.124	-0.128	-0.131	-0.135	-0.139
$B_2 = -0.00386$	$n_2+A_2+nB_2$		+0.098				+4.634				+3.099				+2.783	+2.429	+1.156	+1.892	-0.012
	Residuals		+0.076				-0.064				+0.097				+0.063	-0.021	-0.035	+0.045	-0.014
$A_3 = +0.049$	n_3		-0.18					+5.02					+0.08	-0.17	+2.51	+2.28	+1.13	+1.82	
3	$+A_3$		+0.049					+0.049					+0.049	+0.049	+0.049	+0.049	+0.049	+0.049	
	$+nB_3$		+0.068					+0.085					+0.103	+0.106	+0.109	+0.113	+0.131	+0.120	
$B_3 = +0.00342$	$n_3+A_3+nB_3$		-0.063					+5.154					+0.272	-0.015	+2.668	+2.442	+1.156	+1.989	
	Residuals		-0.077										+0.113	-0.233	-0.052	-0.008	-0.035	+0.142	
$A_4 = -0.127$	n_4			+3.37							+3.01	+0.65	+0.28	+0.48	+2.84	+2.57	+1.18	+1.82	
4	$+A_4$			-0.127							-0.127	-0.127	-0.127	-0.127	-0.127	-0.127	-0.127	-0.127	
	$+nB_4$			+0.016							+0.022	+0.022	+0.023	+0.024	+0.025	+0.026	+0.026	+0.027	
$B_4 = +0.00077$	$n_4+A_4+nB_4$			+3.259							+2.905	+0.545	+0.176	+0.377	+2.738	+2.469	+1.079	+1.720	
	Residuals			-0.021							-0.097	+0.135	+0.017	+0.159	+0.018	+0.019	-0.112	-0.127	
$A_5 = +0.225$	n_5			+3.32					+7.03					+0.47	+2.90	+2.65	+1.37	+2.01	0
5	$+A_5$			+0.225					+0.225					+0.225	+0.225	+0.225	+0.225	+0.225	+0.225
	$+nB_5$			-0.243					-0.301					-0.359	-0.370	-0.380	-0.393	-0.405	-0.417
$B_5 = -0.01157$	$n_5+A_5+nB_5$			+3.302					+6.954					+0.336	+2.755	+2.493	+1.202	+1.830	-0.192
	Residuals			+0.022										+0.118	+0.035	+0.043	+0.011	-0.017	-0.194
$A_6 = +0.121$	n_6				+2.74							+0.15	-0.16	+0.15	+2.60	+2.33	+1.11	+1.71	0
6	$+A_6$				+0.121							+0.121	+0.121	+0.121	+0.121	+0.121	+0.121	+0.121	+0.121
	$+nB_6$				+0.003							+0.005	+0.005	+0.005	+0.005	+0.005	+0.005	+0.006	+0.006
$B_6 = +0.00016$	$n_6+A_6+nB_6$				+2.864							+0.276	-0.034	+0.276	+2.726	+2.456	+1.236	+1.837	+0.127
	Residuals											-0.134	-0.193	+0.058	+0.006	+0.006	+0.045	-0.010	+0.125
	$\sum_{C_n}(n_N+A_N+nB_N)$		+0.027	+6.561	+2.864		+9.395	+5.154	+6.954		+6.004	+0.821	+0.637	+1.091	+16.321	+14.703	+7.146	+11.080	+0.001
	C_n		2	2	1		2	1	1		2	2	4	5	6	6	6	6	4
	$n_0 = \frac{1}{C_n}\sum_{C_n}(n_N+A_N+nB_N)$		+0.014	+3.280	+2.864		+4.698	+5.154	+6.954		+3.002	+0.410	+0.159	+0.218	+2.720	+2.450	+1.191	+1.847	0

This time, however, approximate values of A_1, etc., are to hand for substitution in these equations, and since these A terms are already small, the evaluation of the six quantities $B_1 \ldots B_6$ from the resulting equations has in this case needed no correction. At the appropriate stage the second equation of condition

$$0 = \sum_{C_{36}} A_N + 36 \sum_{C_{36}} B_N$$

is introduced. Since the calibrations involving the end point 36 are Nos. 1, 2, 5 and 6, and in all of these the error of the point has been put = 0, this equation becomes

$$0 = A_1 + A_2 + A_5 + A_6 + 36 (B_1 + B_2 + B_5 + B_6).$$

With the six values of the B's a return is made to the first step in the calculation of the A's. The terms in B instead of being neglected now have the values of B substituted in them, thus giving new values, placed underneath the originals, for the right-hand sides of the equations in A. The right-hand sides of these are modified throughout in the manner shown by the (small) numbers underneath the originals and lead to the second approximations shown in the small rectangular table underneath the first approximations. A check back on the original determinant now showed that the values of the A's and B's at this stage were good enough to give the quantities $A_N + nB_N$ to within about $0 \cdot 001\,\mu$. They were therefore accepted as final.

In Table XIV the final values $1_0, 2_0 \ldots n_0$ are obtained by means of equation (1) :—

$$n_0 = \frac{1}{C_n} \sum_{C_n} (n_N + A_N + nB_N).$$

For each calibration the separate quantities $n_N + A_N + nB_N$ are evaluated. These are the figures to be compared with the figures n_0 in order to obtain residuals (observed — calculated), for they represent the tilting of the original graph for a particular calibration into the best position for obtaining mutual agreement, while the indicated relative position of the points on the bar is still exactly that given by the observations in this calibration.

In the table, the residuals are placed under the quantities $n_N + A_N + nB_N$. They are 95 in number, and are with one exception $< 0 \cdot 2\,\mu$. The root mean square value of a residual is $0 \cdot 076\,\mu$ which shows considerable improvement on the root mean square value $0 \cdot 131\,\mu$ of the quantities $n_N - \bar{n}$, *i.e.*, the root mean square residual previous to "least square" adjustment.

TABLE XIII.—Normal Equations—Summarised Solution.

Approximate

$A_1 = -0.2462$
$A_2 = +0.1223$
$A_3 = +0.0491$ [illegible]
$A_4 = -0.2251$
$A_5 = +0.3201$ [illegible]
$A_6 = +0.154$ [illegible]

2nd Approximation

$A_1 = -0.2483$
$A_2 = +0.1226$
$A_3 = +0.0499$
$A_4 = -0.1275$
$A_5 = +0.3281$
$A_6 = +0.1266$

Approximate elimination of B's

B_1 [illegible]
$B_2 = 0.5566$ [illegible]
$B_3 = -0.3117$ [illegible]
$B_4 = -0.0176$ [illegible]
$B_5 = $ [illegible]
$B_6 = -0.2158$ [illegible]

[To face p. 313.]